A grand canonical approach to linear Lanford theorem near equilibrium

Master thesis

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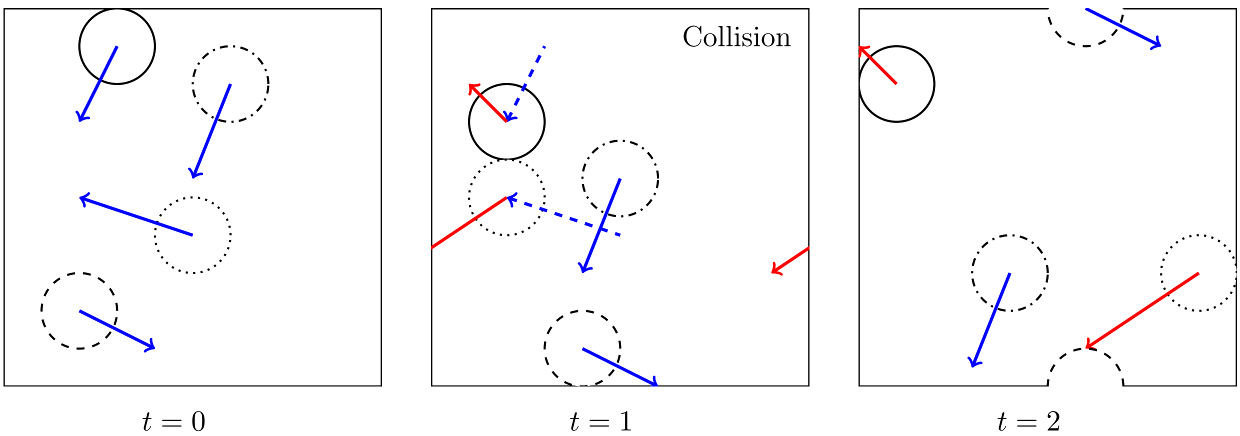

Figure 1: Example of microscopic dynamics on the torus

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1 Introduction

The Boltzmann equation, introduced in 1872 by the physicist who gave it his name, offers a model for rarefied gases dynamics, in the case of very low densities. The solutions to this equation irreversibly converge in large times to a well-known equilibrium depending only on the energy of the system. However, the state of the gas at a microscopic scale, from which the Boltzmann model is derived in the Boltzmann-Grad limit, is given by classical Newton equations, even though the solutions to such equations are completely time riversible. This seeming paradox led Boltzmann's contemporaries into doubting the validity of his model for a very long time.

Nevertheless, time and experience have proven Boltzmann right, and his work on entropy is now at the core of statistical physics. The rigorous derivation of the mesoscopic Boltzmann equation from microscopic Newton equations has even been mathematically proved in 1975 by Oscar Erasmus Lanford III in the case of the hard sphere model, but the methods he used suffer from a strong ridigity that hinders to prove his result for long time scales. In fact, his proof is only valid for very small times, when only about a fifth of particles have collided. The major obstruction that justifies such a limitation is the correlation that happens when two particles collide: the system loses some of its chaotic properties, making it very hard to deal with any recollision of particles.

Nevertheless, for a linear version of the Boltzmann equation describing the behavior of a finite number of tagged particles near equilibrium, Henk van Beijeren, Lanford, Joel Louis Lebowitz and Herbert Spohn showed in 1980 a convergence for large time scales. A few decades later, in 2013, Isabelle Gallagher, Laure Saint-Raymond and Benjamin Texier reopened Lanford's work in a big paper filling some gaps in the articles of Lanford and his former student Francis Gordon King, including the case of positive short-range potentials, and hence providing precise estimates on the convergence rate. Eventually, this work has led in 2014 to an article by Thierry Bodineau, Gallagher and Saint-Raymond on the convergence rate in the linear case, and its application to hydrodynamic limits from the Boltzmann equation, especially to the apparition of a rescaled deterministic behavior of the tagged particle converging in law to a Brownian motion.

In the present paper, we consider the proof of the latter article, fixing some errors and adapting it in a different formalism called grand canonical ensemble, as opposed to the canonical ensemble in which the latter article is written. This formalism demands the introduction of a little bit more of definitions, and changes the way some propositions are proved, but simplifies the calculus afterwards. Indeed, it corresponds to relaxing the condition on the number of particles, making it a random number, so that the considered objects will be more easily comparable to the limiting Boltzmann equation describing a very large amount of particles.

We start presenting the microscopic model governed by Newton laws, before introducing the new formalism of the grand canonical ensemble. After that, we expound the linear model near equilibrium and its motivation to finally state, and then prove, the main theorem. This theorem yields a large time convergence rate for the convergence of the microscopic linear model to Boltzmann equilibrium, in the grand canonical formalism. Some probabilistic consequences of this theorem, motivating the need of an explicit convergence rate so as to go to hydrodynamic limits, are also stated – yet not proven – along with the main theorem. A few lines in the end of the document are dedicated to an opening to further studies, and some side results are stated in an appendix.

2 Microscopic modelization: Newton and Liouville equations

The results we present in this paper are only valid in a certain scaling, which corresponds to the framework of the Boltzmann model, called *Boltzmann–Grad* scaling. This scaling supposes that the *mean free path* of the particles, i.e. the average distance that a particle may hope to go freely between two collisions, remains of order one. Hence, a free particle must go through a tube of the same order as its section – that is ε^{d-1} if ε denotes the diameter of the spheres – before encountering one of the other $N-1$ particles, which yields the following scaling:

$$N\varepsilon^{d-1} = O(1). \tag{1}$$

One can see that the proportion of volume occupied by the spheres is of order ε and hence goes to zero: we talk about *rarefied gases*, or *low-density* scaling.

In this section, we start by describing the Newton model at a microscopic scale, followed by its Liouville version and the ensuing hierarchy satisfied by the marginals of the studied densities.

2.1 Newton equations

To modelize a gas of N interacting particles evolving on the d-dimentional unit torus, we denote $\underline{x}_N = (x_1, \dots, x_N) \in \mathbb{T}^{dN}$ their positions and $\underline{v}_N = (v_1, \dots, v_N) \in \mathbb{R}^{dN}$ their velocities. The phase space describing a single particle is thus given by $\mathcal{D} = \mathbb{T}^d \times \mathbb{R}^d$, and we denote $\underline{z}_N = (x_1, v_1, \dots, x_N, v_N) \in \mathcal{D}^N$ the state of the system.

In the case of hard spheres, we impose the following restriction on the domain

$$\mathcal{D}_N^\varepsilon = \{\underline{z}_N \in \mathcal{D}^N;\ \forall\, i \neq j,\ |x_i - x_j| > \varepsilon\}, \tag{2}$$

so that the particles behave like spheres of diameters ε which cannot overlap, justifying the name for the model of *hard spheres*.

Within the open domain $\mathcal{D}_N^\varepsilon$, the particles does not meet and hence their coordinates simply follow Newton equations for uniform line movement, that are

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = v_i, \qquad \frac{\mathrm{d}v_i}{\mathrm{d}t} = 0. \tag{3}$$

At the contrary, on the boundary of $\mathcal{D}_N^\varepsilon$, at least two particles have gone into contact. For the collision of the pair of particles (i,j), we hence have $|x_i - x_j| = \varepsilon$. In our model, the interaction is instantaneous. Denoting (v_i, v_j) the pre-collisional velocities and $({v_i}', {v_j}')$ the post-collisional velocities, the movement is given by the four following physical rules, for particles of identical mass, in the model of *elastic collisions*:

- movement is contained in a plane: the problem is $2d$−dimensional (4 scalar unknowns)
- force is collinear to $x_i - x_j$, and so is ${v_k}' - v_k$, $k \in \{i,j\}$ (1 scalar equation)
- *momentum* is conserved: $v_i + v_j = {v_i}' + {v_j}'$ (2 scalar equations)
- *kinetic energy* is conserved: $|v_i|^2 + |v_j|^2 = |{v_i}'|^2 + |{v_j}'|^2$ (1 scalar equation)

From the three first equations, we get that for a certain $\lambda \in \mathbb{R}$ we have

$$\begin{cases} {v_i}' = v_i + \lambda(x_i - x_j) \\ {v_j}' = v_j - \lambda(x_i - x_j), \end{cases} \tag{4}$$

and taking the squared norms in these equations before adding both of them to make appear kinetic energy, recalling that $|x_i - x_j| = \varepsilon$, we get the expression of $\lambda \neq 0$, that provides the following relation between pre- and post-velocities, describing all the collisions

$$\begin{cases} {v_i}' = v_i - \dfrac{1}{\varepsilon^2}\left[(v_i - v_j)\cdot(x_i - x_j)\right](x_i - x_j) \\ \\ {v_j}' = v_j + \dfrac{1}{\varepsilon^2}\left[(v_i - v_j)\cdot(x_i - x_j)\right](x_i - x_j). \end{cases} \tag{5}$$

Figure 2 illustrates two examples of velocities variation due to a collision between two particles,

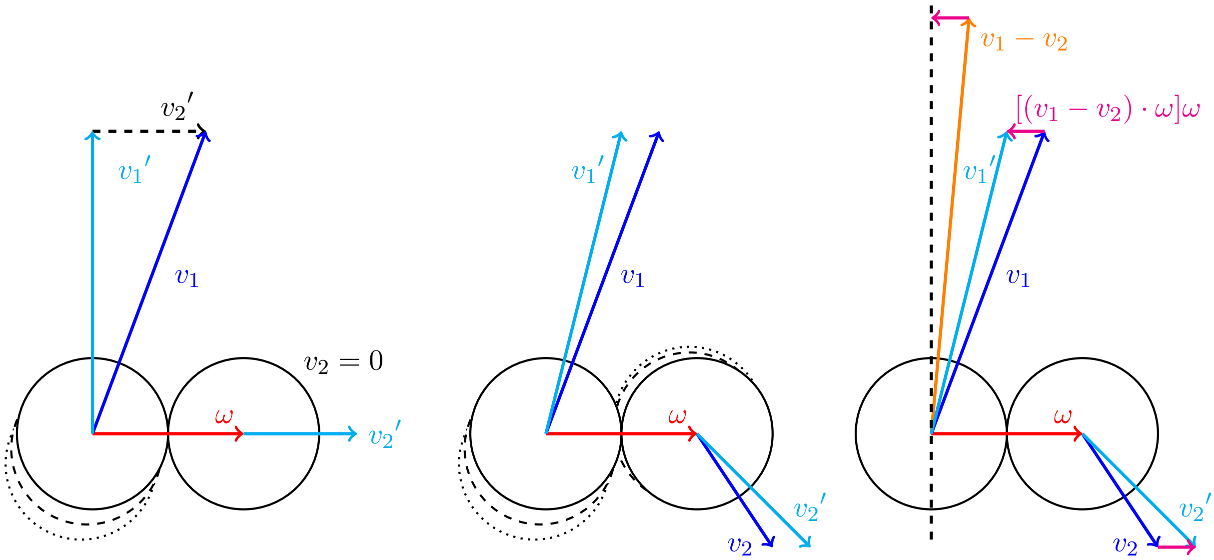


Figure 2: Geometric drawing of pre- and post-velocities

according to system (5) and with the same notation: the first sketch in the situation (always true up to a change of reference frame) where $v_2 = 0$, and the two last ones in a more general case, the third one being the detailed explanation of the second one.

Throughout this document, we will use the notation

$$\omega = \frac{x_i - x_j}{\varepsilon}, \tag{6}$$

and the following identity derived directly from (5)

$$\omega \cdot ({v_i}' - {v_j}') = -\omega \cdot (v_i - v_j) \leq 0. \tag{7}$$

In particular, one may check that the map $(v_i, v_j, \omega) \mapsto ({v_i}'(\omega), {v_j}'(\omega), \omega)$ is easily invertible, and that denoting $({v_i}^*, {v_j}^*)$ the *pre*-collisional velocities given by some post-collisional velocities (v_i, v_j), we have

$$({v_i}'(\omega), {v_j}'(\omega)) = ({v_i}^*(-\omega), {v_j}^*(-\omega)); \tag{8}$$

this map is almost an involution, up to change the sign of ω so as to be back to the pre-collisional case. In practice, since the sign of ω is squared in the formulas, they stay the same.

These equations well define a dynamics for the gas of particles, as soon as the trajectories stay away from collisions of three or more particles (in the case where $|x_i - x_j| = |x_j - x_k| = \varepsilon$) and of infinite amounts of collisions happening in finite times. This question is completely treated in [5] whose authors show that the set of initial configurations leading to these pathological cases is of measure zero, reusing the geometric ideas of [1]. Finally, one can observe that these equations define a perfectly time-reversible dynamics, of which Figure 1 (p. 2) pictures an example in the case of four interacting particles.

Figure 3 gives an insight of the global trajectories by picturing the positions slightly before and after the collision. This kind of collision may as well describe the behavior of billiards balls.

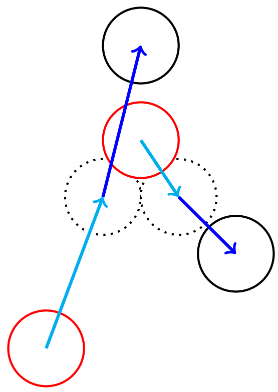


Figure 3: Positions slightly before (in red) and after (in black) the collision

2.2 Liouville equation and BBGKY hierarchy

We now consider a probabilistic description of the gas, assuming that we cannot determine for sure the positions and velocities of the particles, but that we may only have access to them through the probability density in phase space f_N of the random variable $\underline{Z}_N$ on $\mathcal{D}_N^\varepsilon$, representing the infinitesimal probability of configurations of particles. As the particles are supposed indistinguishable, f_N must be invariant by exchange of particles, i.e.

$$\forall\ 1 \leq i < j \leq N,\ f_N(\dots, z_i, \dots, z_j, \dots) = f_N(\dots, z_j, \dots, z_i, \dots). \tag{9}$$

Newton equations in the open domain provides the following *Liouville equation* on the density:

$$\partial_t f_N + v_N \cdot \nabla_{X_N} f_N = 0 \ \text{ in } \ \mathcal{D}_N^\varepsilon. \tag{10}$$

At the boundary, for the collision of a pair of particles (i, j), the density in naturally defined by the above transport Liouville equation (10) in the case of pre-collisional states. Nevertheless, denoting

$$\underline{z}_N^* = (z_1, \dots, x_i, {v_i}^*, \dots, x_j, {v_j}^*, \dots, z_N)$$

the pre-collisional state associated with a post-collisional state $\underline{z}_N$, we have the following boundary condition on the density for *outgoing* particles (i.e. entering the domain $\mathcal{D}_N^\varepsilon$, going out of collision)

$$|x_i - x_j| = \varepsilon \text{ and } (x_i - x_j) \cdot (v_i - v_j) > 0 \qquad \Rightarrow \qquad f_N(\underline{z}_N) = f_N(\underline{z}_N^*). \tag{11}$$

As the number N of particles will be going to infinity, we will consider the *marginals* of f_N, which remain in the same functional spaces as N varies. These marginals are linked one with another through the *BBGKY hierarchy*, called after Nikolaï Bogolioubov (Николай Николаевич Боголюбов), Max Born, Herbert Green, John Kirkwood and Jacques Yvon. The following calculus is geometric and analytic considerations that eventually lead to the aforementioned hierarchy (18). The reader is invited to skip them if pleased to do so.

The marginals of f_N are naturally defined as follows for $s \leq N$

$$f_N^{(s)}(t, \underline{z}_s) = \int_{\mathbb{R}^{d(N-s)}} f_N(t, \underline{z}_s, z_{s+1}, \dots, z_N) \mathrm{d}z_{s+1} \dots \mathrm{d}z_N. \tag{12}$$

To find a relation between them, let us consider an observable $\psi \in \mathcal{C}_c^\infty(\mathcal{D}_s^\varepsilon)$ that satisfies the same exchange (9) and boundary (11) conditions as f_N. Integrating the Liouville equation (10) multiplied by $\psi(t, \underline{z}_s)\mathbb{1}_{\underline{z}_N \in \mathcal{D}_s^\varepsilon}$, the first term simply becomes by integration by parts and the definition of the s-th marginal (12)

$$
\begin{aligned}
&\int \partial_t f_N\left(t, \underline{z}_N\right) \psi(t, \underline{z}_s)\mathbb{1}_{\underline{z}_N \in \mathcal{D}_s^\varepsilon} \mathrm{d}\underline{z}_N \mathrm{d}t \\
&= -\int_{\mathbb{R}^{2dN}} f_N^{(s)}\left(0, \underline{z}_s\right) \psi(0, \underline{z}_s) \mathrm{d}\underline{z}_s - \int_{\mathbb{R}^+ \times \mathbb{R}^{2dN}} f_N^{(s)}(t, \underline{z}_s) \partial_t \psi(t, \underline{z}_s) \mathrm{d}\underline{z}_s \mathrm{d}t \\
&= \int_{\mathbb{R}^+ \times \mathbb{R}^{2dN}} \partial_t f_N^{(s)}(t, \underline{z}_s)\psi(t, \underline{z}_s) \mathrm{d}\underline{z}_s \mathrm{d}t.
\end{aligned} \tag{13}
$$

The second term gives by Green formula in space

$$
\begin{aligned}
&\sum_{i=1}^N \int_{\mathcal{D}_N^\varepsilon} v_i \cdot \nabla_{x_i} f_N(\underline{z}_N)\psi(\underline{z}_s) \mathrm{d}\underline{z}_N \\
&\quad = \int_{\mathcal{D}_N^\varepsilon} \mathrm{div}_{\underline{x}_N}(\underline{v}_N f_N(\underline{z}_N))\psi(\underline{z}_s) \mathrm{d}\underline{z}_N \\
&\quad = -\sum_{i=1}^N \int_{\mathcal{D}_N^\varepsilon} f_N(\underline{z}_N) v_i \cdot \nabla_{x_i}\psi(\underline{z}_s) \mathrm{d}\underline{z}_N \\
&\qquad + \sum_{1 \le i < j \le N} \int_{\mathbb{R}^{dN}} \int_{\Sigma(i,j)} n_{i,j} \cdot \underline{v}_N f_N(\underline{z}_N)\psi(\underline{z}_s) d\sigma^{i,j}(\underline{x}_N) \mathrm{d}\underline{v}_N
\end{aligned}
$$

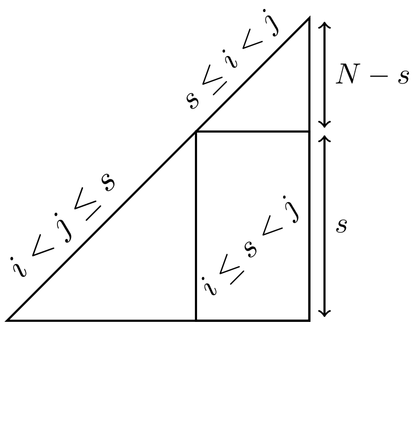
since the triple collisions are of measure zero, and using the following notation to split the double collisions according to the pair of particles involved

$$
\Sigma(i,j) = \{\underline{z}_N \in \mathbb{R}^{dN};\ |x_i - x_j| = \varepsilon\}.
$$

Hence, as $n_{i,j} = \dfrac{1}{\sqrt{2}\varepsilon}(0, \ldots, x_j - x_i, \ldots, x_i - x_j, \ldots, 0)$, the double sum can be rewritten

$$
\sum_{1 \le i < j \le N} \int_{\mathbb{R}^{dN}} \int_{\Sigma(i,j)} \frac{(x_j - x_i) \cdot (v_i - v_j)}{\sqrt{2}\varepsilon} f_N(\underline{z}_N)\psi(\underline{z}_s) \mathrm{d}\sigma^{i,j}(\underline{x}_N) \mathrm{d}\underline{v}_N. \tag{14}
$$

We henceforth split this sum according to the following figure.



For $1 \leq i \leq s < j \leq N$, using the change of variable developped in Appendix 9.1,

$$\begin{aligned}
&\int_{\mathbb{R}^{dN}} \int_{\Sigma(i,j)} \frac{(x_j - x_i)\cdot(v_i - v_j)}{\sqrt{2}\varepsilon} f_N(\underline{z}_N)\psi(\underline{z}_s)\mathrm{d}\sigma^{i,j}(\underline{x}_N)\mathrm{d}\underline{v}_N \\
&= \int_{\mathbb{R}^{dN}} \int_{\mathbb{R}^{d(N-1)}} \int_{\mathbb{S}(x_i,\varepsilon)} \frac{(x_j - x_i)}{\varepsilon}\cdot(v_i - v_j) f_N(\underline{z}_N)\psi(\underline{z}_s)\mathrm{d}x_1 \dots \mathrm{d}\omega^i(x_j)\dots \mathrm{d}x_N \mathrm{d}\underline{v}_N \qquad (15)\\
&= \varepsilon^{d-1} \int_{\mathbb{R}^{d(s+1)}} \int_{\mathbb{R}^{ds}} \int_{\mathbb{S}^{d-1}} \omega\cdot(v_i - v_{s+1}) f_N^{(s+1)}(\underline{z}_s, x_i + \varepsilon\omega, v_{s+1})\psi(\underline{z}_s)\mathrm{d}\underline{x}_s \mathrm{d}\omega \mathrm{d}\underline{v}_{s+1}
\end{aligned}$$

where we parametrized $x_j = x_i + \varepsilon\omega \Leftrightarrow \omega = (x_j - x_i)/\varepsilon$, using the exchangeability to swap x_j and x_{s+1}, and eventually the definition of the $(s+1)$-th marginal.

Otherwise, if $i, j \in [\![1, s]\!]$ or $i, j \in [\![s+1, N]\!]$, let us fix $\underline{x}_N$ such that $\underline{z}_N \in \Sigma(i,j)$. Denoting

$$\underline{v}_N^* = (v_1, \dots, {v_i}^*, \dots, {v_j}^*, \dots, v_N)$$

we have by the boundary condition (11) on f_N and ψ, using the relation (7) between pre- and post-velocities and the change of variable $(v_i, v_j) \mapsto ({v_i}^*, {v_j}^*)$ of Jacobian 1 (cf. Appendix 9.2),

$$\begin{aligned}
&\int_{\mathbb{R}^{dN}} \omega\cdot(v_i - v_j) f_N(\underline{z}_N)\psi(\underline{z}_s)\mathrm{d}\underline{v}_N \\
&= \int [\omega\cdot(v_i - v_j)]_+ f_N(\underline{z}_N^*)\psi(\underline{z}_s^*)\mathrm{d}\underline{v}_N - \int [\omega\cdot(v_i - v_j)]_- f_N(\underline{z}_N)\psi(\underline{z}_s)\mathrm{d}\underline{v}_N \\
&= \int [-\omega\cdot({v_i}^* - {v_j}^*)]_+ f_N(\underline{z}_N^*)\psi(\underline{z}_s^*)\mathrm{d}\underline{v}_N^* - \int [\omega\cdot(v_i - v_j)]_- f_N(\underline{z}_N)\psi(\underline{z}_s)\mathrm{d}\underline{v}_N = 0.
\end{aligned}$$

Hence, the only contributing terms of the double sum are the crossed terms calculated in (15), which at a fixed i appear identically $(N-s)$ times each one by symmetry, so that the double sum is equal to

$$-(N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \omega\cdot(v_{s+1} - v_i) f_N^{(s+1)}(\underline{z}_s, x_i + \varepsilon\omega, v_{s+1})\mathrm{d}\omega \mathrm{d}v_{s+1}. \qquad (16)$$

Henceforth, we introduce the operator

$$\widetilde{\mathcal{C}}_s f_N^{(s+1)}(\underline{z}_s) \doteq (N-s)\varepsilon^{d-1} \sum_{i=1}^{s} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \omega\cdot(v_{s+1} - v_i) f_N^{(s+1)}(\underline{z}_s, x_i + \varepsilon\omega, v_{s+1})\mathrm{d}\omega \mathrm{d}v_{s+1}, \qquad (17)$$

so that our calculus eventually leads to the following BBGKY hierarchy on the marginals of f_N

$$\left(\partial_t f_N^{(s)} + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f_N^{(s)}\right)(t, \underline{z}_s) = \widetilde{\mathcal{C}}_s f_N^{(s+1)}(t, \underline{z}_s). \qquad (18)$$

3 Grand canonical formalism

In this section, we develop a new formalism, called *grand canonical* formalism, or in physics grand canonical ensemble, as opposed to the previous formalism, wherein the number N of particles was deterministically fixed, called canonical formalism. Hence, in this new formalism, the number of particles will become a random variable. Indeed in statistical physics, the very exact number of particles is extremely hard to determine. One might also imagine a system that is not completely closed but may sometimes exchange particles with a large reservoir. Both these situations justify the introduction of the grand canonical formalism, less rigid, which in various cases brings a foremost stochastic simplification in calculus, which will happen in this paper, and sometimes yields explicit results that are hindered in the canonical case. For curiosity, to provide an instance of such a case, in the field of quantum mechanics the density matrix associated to the free Bose gas in Fock space is explicit, though in canonical ensemble it is very impractical to manipulate (see the course by Jan Philip Solovej [10, section 7]).

3.1 Chemical potential and correlation functions

Henceforth, the number of particles $\mathcal{N}$ will be chosen close to a Poisson random variable of parameter μ_ε, with the scaling condition of rarefied gases (as discussed in the introduction of section 2): at fixed[1] $\alpha > 1$,

$$\mu_\varepsilon \varepsilon^{d-1} = \alpha. \tag{19}$$

The considered domain is thus the disjoint reunion of all possible N-particle domains:

$$\mathcal{D}^\varepsilon = \coprod_{N \in \mathbb{N}} \mathcal{D}^\varepsilon_N. \tag{20}$$

Let us remark that in quantum mechanics, as the domain of admissible system states (a subspace of $\mathbb{L}^2$) is endowed with a vector structure, the grand-canonical domain that we consider is often the Fock space $\bigoplus \mathcal{D}^\varepsilon_N$. Such a vector structure would make no sense here.

Initial data is supposed completely chaotic, which means that at time $t = 0$, conditionally to the event $(\mathcal{N} = N)$ for $N \in \mathbb{N}$, and conditionally to the hard-sphere property (that can be written $\underline{z}_N \in \mathcal{D}^\varepsilon_N$) the initial probability density of the random state of the system $\underline{Z}_N$ in $\mathcal{D}^N$ is the tensor product of uncorrelated identical densities $f_0(x, v)$ on $\mathcal{D}$, that will be explicitly close to equilibrium in our linear case. More generally, they are usually chosen such that for some $\beta_0 > 0$,

$$|f_0(x, v)| + |\nabla_x f_0(x, v)| \le C_0 \exp\left(-\frac{\beta_0}{2}|v|^2\right). \tag{21}$$

More precisely, the initial probability density of the random state of the system $\underline{Z}_\mathcal{N}$ in $\mathcal{D}^\varepsilon$ is taken as such

$$p^0(\underline{z}) = \frac{1}{\mathcal{Z}} \sum_{N \ge 0} \mathbb{1}_{\mathcal{D}^\varepsilon_n}(\underline{z}) \frac{\mu^n}{n!} \prod_{i=1}^n f_0(z_i), \tag{22}$$

where $\mathcal{Z}$ is the following constant of normalization, called *grand canonical partition function*

$$\mathcal{Z} = 1 + \sum_{n \ge 1} \frac{\mu^n}{n!} \int_{\mathcal{D}^\varepsilon_n} \prod_{i=1}^n f_0(z_i) \, \mathrm{d}\underline{z}_n. \tag{23}$$

Formula (22) is valid with the convention $\int_{\mathcal{D}^\varepsilon_0} \mathrm{d}\underline{z}_0 = 1$.

We will henceforth study the *projected densities*, defined on $\mathcal{D}^\varepsilon_N$ at time $t = 0$ by

$$W^0_N(\underline{z}_N) = \frac{1}{\mathcal{Z}} \mu^N \prod_{i=1}^N f_0(z_i) \mathbb{1}_{\mathcal{D}^\varepsilon_N}(\underline{z}_N), \tag{24}$$

and at time $t \ge 0$, denoted $W_N(t)$, by the pushed-forward densities of W^0_N by the deterministic flow associated to the hard-sphere dynamics – well-defined up to a zero-measure set as discussed in the study of Newton gas dynamics. Hence, the sequence $(W_N(t))_{N\in\mathbb{N}}$ is encrypting the whole available

[1] This formal choice of scaling is useful for following studies such as hydrodynamic limits departing from Boltzman equation, in the limit $\alpha \longrightarrow \infty$ (see Theorem 5.2 and more generally [9]).

information on the state of the system at time $t \geq 0$. More precisely, for $A \subset \mathbb{N}$ and $B \subset \mathcal{D}^\varepsilon$, we know that the probability that the random couple $(\mathcal{N}, \underline{Z}_\mathcal{N})$ belongs to the cylinder $A \times B$ is given by

$$\mathbb{P}\left[(\mathcal{N}, \underline{Z}_\mathcal{N}) \in A \times B\right] = \sum_{n \geq 0} \frac{\mathbb{1}_A(n)}{n!} \int_{B \cap \mathcal{D}^n} W_n(t, \underline{z}_n) \mathrm{d}\underline{z}_n. \tag{25}$$

Like when we have studied marginals in the previous section so as to fix the domain on which studied functions were defined, while having information on all possible configurations, we now introduce the *correlation functions*

$$F_n^0(\underline{z}_n) = \mu^{-n} \sum_{p \geq 0} \frac{1}{p!} \int_{\mathcal{D}^p} W_{n+p}^0(\underline{z}_n, z_{n+1}, \ldots, z_{n+p}) \mathrm{d}z_{n+1} \ldots \mathrm{d}z_{n+p}. \tag{26}$$

For any time $t \geq 0$, we define similarly (in a more condensed way)

$$F_n(t) = \mu^{-n} \sum_{p \geq 0} \frac{1}{p!} W_{n+p}^{(n)}(t). \tag{27}$$

Physically, we can see in this formula that we average all the n-th marginals for all admissible number of particles ($\mathcal{N} \geq n$). The normalizing factor $p!$ is linked to the Poisson law approximating the distribution of $\mathcal{N}$, but in this formula it can be understood as a combinatory coefficient, for we add p particles without paying attention to their order. Note that F_n is not normalized as a real probability density, to simplify greatly the calculus, but a study of partition functions (in the same way as in Appendix 9.3) proves that its total weight goes to 1 as μ goes to infinity in the Boltzmann-Grad scaling.

Both sequences $(F_n)_n$ and $(W_N)_N$ contain the same information on the state of the system. Indeed it can be checked that they are linked by the inversion formula

$$W_n(\underline{z}_n) = \mu^n \sum_{p=0}^{\infty} \frac{(-\mu)^p}{p!} \int_{\mathcal{D}^p} F_{n+p}(\underline{z}_{n+p}) \mathrm{d}z_{n+1} \ldots \mathrm{d}z_{n+p}, \tag{28}$$

developing the sums and using the following combinatory formula: for any $s \geq 1$,

$$s! \sum_{p=0}^{s} \frac{(-1)^p}{p!(s-p)!} = (1-1)^s = 0. \tag{29}$$

One may remark that the *law of* $\mathcal{N}$ stems from formula (25) : indeed, for any $N \in \mathbb{N}$,

$$\mathbb{P}\left[\mathcal{N} = N\right] = \frac{1}{\mathcal{Z}} \frac{\mu^N}{N!} \int_{\mathcal{D}^N} \prod_{i=1}^{N} f_0(z_i) \mathbb{1}_{\mathcal{D}_N^\varepsilon}(\underline{z}_n) \mathrm{d}\underline{z}_n. \tag{30}$$

In our following study, the initial density will be chosen as a space-uniform equilibrium depending only on the velocities, so that the equation above will become

$$\mathbb{P}\left[\mathcal{N} = N\right] = \frac{\mathcal{Z}_N^c}{\mathcal{Z}} \frac{\mu^N}{N!}, \tag{31}$$

where $\mathcal{Z}_N^c$ is the *canonical partition function*

$$\mathcal{Z}_N^c = \int_{\mathcal{D}^N} \mathbb{1}_{\mathcal{D}_N^\varepsilon}(\underline{z}_n) \mathrm{d}\underline{z}_n. \tag{32}$$

Since at fixed ε the sequence $(\mathcal{Z}_N^c)_N$ is decreasing until becoming stationary at zero, the random variable $\mathcal{N}$ weights more the lower values than a Poisson law of parameter μ. Appendix 9.3 provides arguments to justify that as μ goes to infinity, the law of $\mathcal{N}$ gets close to a Poisson law of parameter μ, by studying the asymptotic behaviors of the partition functions.

For an observable $h : \mathcal{D}^n \to \mathbb{R}$ and a random particle configuration $(Z_i)_{1\leq i\leq \mathcal{N}}$ at time $t \geq 0$, we may compute the expectation of h under the empirical measure associated to the configuration $(Z_i)_{1\leq i\leq \mathcal{N}}$ in terms of the correlation functions. Indeed,

$$\begin{aligned}\mathbb{E}\left[\sum_{1\leq i_k\neq i_j\leq \mathcal{N}} h(Z_{i_1},\dots,Z_{i_n})\right] &= \mathbb{E}\left[\delta_{\mathcal{N}\geq n}\frac{\mathcal{N}!}{(\mathcal{N}-n)!}h(Z_1,\dots,Z_n)\right]\\ &= \sum_{p=n}^{\infty}\frac{1}{p!}\frac{p!}{(p-n)!}\int_{\mathcal{D}^p} W_p(t,\underline{z}_p)h(\underline{z}_n)\mathrm{d}\underline{z}_p\\ &= \mu^n\int_{\mathcal{D}^n} F_n(t,\underline{z}_n)h(\underline{z}_n)\mathrm{d}\underline{z}_n. \qquad (33)\end{aligned}$$

In particular, for $h = \tilde{1}$, we get

$$\mathbb{E}[\mathcal{N}] = \mu\int_{\mathcal{D}} F_1(t,z_1)\mathrm{d}z_1 = \frac{\mu}{\mathcal{Z}}\sum_{p=1}^{\infty}\frac{\mu^{p-1}}{(p-1)!}\int f_0^{\otimes p}\mathbb{1}_{\mathcal{D}_p^\varepsilon} \leq \mu.$$

3.2 Grand canonical hierarchy

3.2.1 BBGKY hierarchy

Projected densities W_N are subjected to Liouville equation on $\mathcal{D}_N^\varepsilon$ with collisional boundary conditions, exactly like the densities f_N in section 2.2:

$$\partial_t W_N + \underline{v}_N\cdot\nabla_{\underline{x}_N} W_N = 0 \text{ and } W_N(\underline{z}_N) = W_N(\underline{z}'_N) \text{ on the boundary.} \qquad (34)$$

Hence, the same calculus leads to the same hierarchy

$$\partial_t W_N^{(s)} + \sum_{i=1}^{s} v_i\cdot\nabla_{x_i} W_N^{(s)} = \widetilde{\mathcal{C}}_s W_N^{(s+1)}, \qquad (35)$$

where we recall that

$$\widetilde{\mathcal{C}}_s W_N^{(s+1)}(\underline{z}_s) = (N-s)\varepsilon^{d-1}\sum_{i=1}^{s}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\omega\cdot(v_{s+1}-v_i)W_N^{(s+1)}(\underline{z}_s, x_i+\varepsilon\omega, v_{s+1})\mathrm{d}\omega\mathrm{d}v_{s+1}. \qquad (36)$$

To get rid of the dependency in N we denote

$$\mathcal{C}_s W_N^{(s+1)}(\underline{z}_s) \doteq \sum_{i=1}^{s}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\omega\cdot(v_{s+1}-v_i)W_N^{(s+1)}(\underline{z}_s, x_i+\varepsilon\omega, v_{s+1})\mathrm{d}\omega\mathrm{d}v_{s+1} \qquad (37)$$

so that, using the definition of the correlation function (27) and the equation satisfied by the projected marginals (35)

$$\begin{aligned}\partial_t F_n + \underline{v}_n\cdot\nabla_{\underline{x}_n}F_n &= \frac{1}{\mu^n}\sum_{p\geq 1}\frac{1}{p!}\, p\varepsilon^{d-1}\mathcal{C}_n W_{n+p}^{(n+1)}\\ &= \mu\varepsilon^{d-1}\mathcal{C}_n F_{n+1},\end{aligned}$$

i.e. in the special scaling (19) we have chosen,

$$\partial_t F_n + \underline{v}_n \cdot \nabla_{\underline{x}_n} F_n = \alpha\, \mathcal{C}_n F_{n+1}. \tag{38}$$

A significant simplification occurs from the canonical representation (36) to the grand canonical representation (37): we were able to get rid of the dependency in the number of particles N and hence to find directly the dilute gas scaling without the error in $s\varepsilon^{d-1}$ that we had in the canonical case.

From the definition of the collision operator (37), we can split the integral between pre-collisional and post-collisional cases, according to the sign of the scalar product $\omega\cdot(v_{s+1}-v_i)$, to use the boundary condition (11) so as to consider only pre-collisional states governed by the transport Liouville equation, to eventually write after a change of variable

$$\begin{aligned} C_s g^{(s+1)}(\underline{z}_s) = \sum_{i=1}^{s} \Bigg(&\int \Big[\omega \cdot (v_{s+1} - v_i)\Big]_+ g^{(s+1)}(\dots, x_i, {v_i}^*, \dots, x_i + \varepsilon\omega, {v_{s+1}}^*) \mathrm{d}v_{s+1}\mathrm{d}\omega \\ &- \int \Big[\omega \cdot (v_{s+1} - v_i)\Big]_- g^{(s+1)}(\dots, x_i, v_i, \dots, x_i + \varepsilon\omega, v_{s+1}) \mathrm{d}v_{s+1}\mathrm{d}\omega \Bigg) \\ = \sum_{i=1}^{s} \int \Big[\omega \cdot (v_{s+1} - v_i)\Big]_+ \Big(&g^{(s+1)}(\dots, x_i, {v_i}^*, \dots, x_i + \varepsilon\omega, {v_{s+1}}^*) \\ &- g^{(s+1)}(\dots, x_i, v_i, \dots, x_i - \varepsilon\omega, v_{s+1}) \Big) \mathrm{d}v_{s+1}\mathrm{d}\omega. \end{aligned} \tag{39}$$

Let us observe that in the case where the function $g^{(s+1)}$ is **nonnegative** (which will not always be the case when we will iterate the collision operator that may be negative), the modulus of the collision operator is simply defined as follows by using the formula above

$$\begin{aligned} |C_s| g^{(s+1)}(\underline{z}_s) = \sum_{i=1}^{s} \int \Big[\omega \cdot (v_{s+1} - v_i)\Big]_+ \Big(&g^{(s+1)}(\dots, x_i, {v_i}^*, \dots, x_i + \varepsilon\omega, {v_{s+1}}^*) \\ &+ g^{(s+1)}(\dots, x_i, v_i, \dots, x_i - \varepsilon\omega, v_{s+1}) \Big) \mathrm{d}v_{s+1}\mathrm{d}\omega. \end{aligned} \tag{40}$$

Going back to the PDE satisfied by F_n (38), we will now iterate Duhamel's formula. Denoting Θ_n the free transport operator in $\mathcal{D}_n^\varepsilon$ with specular reflection (see for example the review by C. Villani [13] for details), we get

$$\begin{aligned} F_n(t) &= \Theta_n(t) F_n(0) + \alpha \int_0^t \Theta_n(t - t_1) \mathcal{C}_n F_{n+1}(t_1) \mathrm{d}t_1 \\ &= \sum_{k=0}^{\infty} \alpha^k \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \Theta_n(t - t_1) \mathcal{C}_n \Theta_{n+1}(t_1 - t_2) \mathcal{C}_{n+1} \dots \Theta_{n+k}(t_k) F_{n+k}(0) \mathrm{d}t_1 \dots \mathrm{d}t_k. \end{aligned} \tag{41}$$

Thus, introducing the successive-collision operator

$$Q_{n,n+k}(t) \doteq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \Theta_n(t - t_1) \mathcal{C}_n \Theta_{n+1}(t_1 - t_2) \mathcal{C}_{n+1} \dots \Theta_{n+k}(t_k) \mathrm{d}t_1 \dots \mathrm{d}t_k, \tag{42}$$

the equation eventually becomes

$$F_n(t) = \sum_{k=0}^{\infty} \alpha^k Q_{n,n+k}(t) F_{n+k}(0). \tag{43}$$

The convergence for small times of this series in the right functional space will be discussed later as a consequence of Proposition 6.1.3, providing a continuity estimate for the operators $Q_{n,n+k}$.

3.2.2 Formal limiting hierarchy

Formally, in the limit $\varepsilon \longrightarrow 0$, the free transport operators Θ_n on $\mathcal{D}_n^\varepsilon$ are merely converted into the free transport operators $\Theta_n^{\lim}$ on the whole domain $\mathcal{D}^n$. Furthermore, the collision operators (37) formally become

$$C_s^{\lim} g^{(s+1)}(\underline{z}_s) = \sum_{i=1}^{s} \int \Big[\omega \cdot (v_{s+1} - v_i)\Big]_+ \Big(g^{(s+1)}(\dots, x_i, {v_i}^*, \dots, x_i, {v_{s+1}}^*) - g^{(s+1)}(\dots, x_i, v_i, \dots, x_i, v_{s+1})\Big) \mathrm{d}v_{s+1} \mathrm{d}\omega. \tag{44}$$

We will later study the modulus of this operator, given simply by changing the minus sign into a plus sign, like in (40). These operators define a limiting hierarchy (60): our main goal is to rigorously justify the convergence of the BBGKY hierarchy to the limiting Boltzmann hierarchy.

4 Linear model

Convergence of the BBGKY hierarchy to the limiting hierarchy has been proved by Oscar Erasmus Lanford III in 1975 but with a very small time of validity [6][7]. To be able to get results in a long-time scale, we here introduce a linear model that is restricted to small perturbations around Boltzmann equilibrium.

4.1 Boltzmann general equation

To describe the evolution of the model due to the collisions, we have to consider the gain of post-collisional velocities and the loss of pre-collisional velocities. The general Boltzmann equation on the phase space density $f(t, x, v)$ is given by

$$\partial_t f + v \cdot \nabla_x f = \alpha \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} [f(v')f(v_c') - f(v)f(v_c)] b(v_c - v, \omega) \mathrm{d}v_c \mathrm{d}\omega, \tag{45}$$

where the cross section b, whose hard-sphere version is given by

$$b(v_c - v, \omega) \doteq [(v_c - v) \cdot \omega]_+ \,, \tag{46}$$

weights all the possible couples of pre-collisional velocity v_c and deflection angle ω such that the collision (v, v_c, ω) leads to the velocities $(v', {v_c}')$ (cf. calculus of the BBGKY hierarchy in section 2.2 for the derivation of the cross section in the case of hard spheres).

We define the collision kernel

$$Q(f, g) = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} [f(v')g(v_c') - f(v)g(v_c)] b(v_c - v, \omega) \mathrm{d}v_c \mathrm{d}\omega, \tag{47}$$

modelizing the modification on f due to its collisions with particles distributed according to g.

4.2 Boltzmann equilibria

Boltzmann model is irreversible in time. Indeed, introducing the entropy

$$S(t) = - \int f(t, x, v) \log f(t, x, v) \mathrm{d}x \mathrm{d}v, \tag{48}$$

Ludwig Boltzmann has historically stated [4] in 1896 that $\frac{\mathrm{d}S}{\mathrm{d}t} \geq 0$ (see [13] for a rigorous proof); the entropy is increasing (or decreasing from a physicist's point of view, defining the entropy as $-S$). A lot of ink has been spilled about this statement, and it has brought much discredit upon Boltzmann among his contemporaries, since it illustrates the case of a reversible system at microscopic scale that becomes irreversible at its mesoscopic limit.

The states of equilibrium, corresponding to $\frac{\mathrm{d}S}{\mathrm{d}t} = 0$, are given by velocity-gaussian distributions, called Maxwell distributions. For simplicity we here consider the following equilibrium states, at temperature β^{-1}

$$M_\beta(v) = \left(\frac{\beta}{2\pi}\right)^{d/2} \exp\left(-\frac{\beta}{2}|v|^2\right). \tag{49}$$

For a positive hard sphere radius ε, we thus introduce the following measure, which is invariant for the ε-dynamics (well-defined up to a zero-measure set), for it only depends on the kinetic energy of the system that is preserved by collisions:

$$M_\beta^{N,\varepsilon}(\underline{z}_N) = \frac{1}{\mathcal{Z}_N^c}\left(\frac{\beta}{2\pi}\right)^{Nd/2} \mathbb{1}_{\mathcal{D}_N^\varepsilon}(\underline{z}_N)\prod_{i=1}^N \exp\left(-\frac{\beta}{2}|v_i|^2\right), \tag{50}$$

where the normalization constant (canonical partition function) is simply given by

$$\mathcal{Z}_N^c = |\mathcal{D}_N^\varepsilon| = \int_{\mathcal{D}^N} \mathbb{1}_{\mathcal{D}_N^\varepsilon}(\underline{z}_N)\mathrm{d}\underline{z}_N. \tag{51}$$

4.3 Perturbed equilibrium

We hence choose to slightly disturb this equilibrium in the space variable for a chosen tagged particle of coordinates $z_1 = (x_1, v_1)$. Note that this perturbation might also be generalized to imply several (yet in finite number) tagged particles, or even to happen in the velocity variable, as discussed in [2, sec. 2.4]. Doing so, we destroy particles exchangeability: given ρ a continuous density on $\mathbb{T}^d$, we introduce the perturbed initial distribution

$$f_0^N(\underline{z}_N) = M_\beta^{N,\varepsilon}(\underline{z}_N)\rho(x_1), \tag{52}$$

which remains normalized since (with natural notation, by invariance by translation of the torus)

$$\int f_0^N(\underline{z}_N)\mathrm{d}\underline{z}_N = \frac{1}{\mathcal{Z}_N^c}\int_{\mathbb{T}^{dN}} \rho(x_1)\mathbb{1}_{\mathcal{D}_N^{\varepsilon,x_1}}(x_2,\dots,x_N)\mathrm{d}x_1\dots\mathrm{d}x_N \tag{53}$$

$$= \frac{1}{\mathcal{Z}_N^c}\int_{\mathbb{T}^{dN}} \rho(x_1)\mathbb{1}_{\mathcal{D}_N^{\varepsilon,0}}(x_2,\dots,x_N)\mathrm{d}x_1\dots\mathrm{d}x_N = 1. \tag{54}$$

Thus, the targeted distribution for positive times, consequence of the evolution of the space perturbation ρ, is the solution of the *linear Boltzmann equation* solved classically in $\mathbb{L}^\infty$. This equation consists in replacing the quadratic collision kernel $Q(f,f)$ (47) in Bolzmann equation (45) by a kernel where the background remains at equilibrium. Explicitly, with the initial condition

$$\varphi(0,x,v) = \rho(x), \tag{55}$$

the linear Boltzmann equation on φ is the following

$$\begin{aligned}\partial_t\varphi + v\cdot\nabla_x\varphi &= \alpha\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}[\varphi(v')-\varphi(v)]M_\beta(v_c)\,[\omega\cdot(v_c-v)]_+\,\mathrm{d}v_c\mathrm{d}\omega \\ &= \alpha\ Q(\varphi, M_\beta),\end{aligned} \tag{56}$$

where the deflection angle $\omega \in \mathbb{S}^{d-1}$ corresponds to the direction of the variation vector of velocities, collinear to $x_i - x_j$ according to the discussion in section 2.1.

4.4 Limiting hierarchy

From the Duhamel formula applied to the PDE (56) satisfied by φ, we have

$$\varphi(t,x,v)=\theta(t)\rho(x)+\alpha\int_0^t\theta(t-s)Q(\varphi,M_\beta)(x,v,s)\mathrm{d}s, \tag{57}$$

where θ is the free flow operator on $\mathcal{D}$. Let us then consider the density

$$M_\beta(v)\varphi(t,x,v)=\theta(t)M_\beta(v)\rho(x)+\alpha\int_0^t\theta(t-s)M_\beta(v)Q(\varphi,M_\beta)(x,v,s)\mathrm{d}s. \tag{58}$$

Considering the integrand above, reiterating Duhamel formula and using linearity we may write

$$\begin{aligned}M_\beta Q(\varphi,M_\beta)(s)&=M_\beta Q\left(\theta(s)\rho+\alpha\int_0^s\theta(s-u)Q(\varphi,M_\beta)(u)\mathrm{d}u,M_\beta\right)\\&=M_\beta Q(\theta(s)\rho,M_\beta)+\alpha\int_0^s\theta(s-u)M_\beta Q\Big(Q(\varphi,M_\beta)(u),M_\beta\Big)\mathrm{d}u.\end{aligned} \tag{59}$$

However, recalling the definitions of the collision operator (44) and collision kernel (56), we can write the first term of the sum above in the following way

$$\begin{aligned}M_\beta(v)Q(\theta(s)\rho,M_\beta)(x,v)&=\int[\theta(v')-\theta(v)]\rho(x)M_\beta(v)M_\beta(v_c)\left[\omega\cdot(v_c-v)\right]_+\mathrm{d}v_c\mathrm{d}\omega\\&=\mathcal{C}_1^{\lim}\left[\theta(s)M_\beta^{\otimes2}\rho\right](x,v).\end{aligned}$$

Similarly, the remaining term in (59) becomes

$$\int_0^s\theta(s-u)M_\beta Q\Big(Q(\varphi,M_\beta)(u),M_\beta\Big)\mathrm{d}u=\int_0^s\theta(s-u)\mathcal{C}_1^{\lim}M^{\otimes2}Q(\varphi,M_\beta)(u)\mathrm{d}u,$$

where once again the integrand may be written

$$M^{\otimes2}Q(\varphi,M_\beta)(u)=\mathcal{C}_2^{\lim}\theta(u)M_\beta^{\otimes3}\rho+\alpha\int_0^u\theta(u-\tau)M_\beta^{\otimes2}Q\Big(Q(\varphi,M_\beta)(\tau),M_\beta\Big)\mathrm{d}\tau,$$

noticing that within the sum in the definition (44) of $\mathcal{C}_s^{\lim}$, even for $s\geq1$, all the terms after the first one vanish, for the perturbation concerns only the first particle and because M_β is invariant by collision (recall definition (49) and the conservation of kinetic energy). Hence, iterating this calculus for the remaining term above, we eventually obtain the infinite Duhamel series of φ (readily generalized for $s\geq1$):

$$\forall s\geq1,\ \varphi M_\beta^{\otimes s}=\sum_{n\geq0}\alpha^n\int_{T_n(t)}\theta(t-t_1)\mathcal{C}_s^{\lim}\theta(t_1-t_2)\dots\theta(t_n)M_\beta^{\otimes s+n}\rho\ \mathrm{d}t_1\dots\mathrm{d}t_n, \tag{60}$$

where $T_n(t)$ is the set of successive collision times defined as

$$T_n(t)=\big\{(t_1,\dots,t_n)\in(\mathbb{R}_+)^n,0\leq t_n\leq\dots\leq t_1\leq t\big\}. \tag{61}$$

We henceforth denote

$$Q_{s,s+n}^{\lim}\doteq\int_{T_n(t)}\Theta_s^{\lim}(t-t_1)\mathcal{C}_s^{\lim}\Theta_{s+1}^{\lim}(t_1-t_2)\dots\Theta_{s+n}^{\lim}(t_n)\mathrm{d}t_1\dots\mathrm{d}t_n, \tag{62}$$

where $\Theta_s^{\lim}$ is the free transport operator on the whole domain $\mathcal{D}^s$. Eventually, we can observe that the family $(M_\beta^{\otimes s}\rho)_s$ satisfies the formal limit of BBGKY hierarchy (see. 3.2.2). Like in the BBGKY case, the convergence for small times of such a Duhamel series in the right functional space is a direct consequence of incoming Proposition 6.1.3, providing a continuity estimate for the operators $Q_{s,s+n}^{\lim}$.

5 Linear Lanford theorem and probabilistic consequences

5.1 Linear Lanford theorem in grand canonical formalism

As announced in the introduction, the following theorem is the linear version of Lanford's theorem ([6, general case] and [12, linear case]) in the grand canonical formalism; indeed it provides a convergence result on the correlation functions associated to a system of particles whose chemical potential μ, which stands asymptotically for the average number of particles, goes to infinity. This result, as in the canonical case, yields a convergence for large time scales with an explicit convergence rate.

Theorem 5.1. *Considering previous notation and framework, in the scaling $\mu\varepsilon^{d-1} = \alpha$ and in the limit $\mu \to \infty$, the first tagged correlation function (27) gets very close to the solution of the linear Boltzmann equation (56) modulated in velocities by the equilibrium Maxwell distribution. Precisely, we have for a constant $C_{d,\beta,A}$ the following convergence speed, for any nonnegative time $t \geq 0$:*

$$\|F_1(t,x,v) - M_\beta(v)\varphi(t,x,v)\|_{\mathbb{L}^\infty([0,t]\times\mathcal{D}^d)} \leq C_{d,\beta,A}\left(\frac{(\alpha t)^{A/(A-1)}}{\log\log\mu}\right)^A \|\rho\|_{\mathbb{L}^\infty}. \tag{63}$$

The proof of this theorem is the point of the following three sections (6, 7 and 8), adapted from the proof of [2], made in the canonical case.

5.2 Probabilistic consequences

This theorem has probabilistic consequences on the behavior of the system. The two first ones are well-known: considering a random particle configuration $(Z_i^\varepsilon(t))_{1\leq i\leq\mathcal{N}}$ initially distributed according to the grand canonical initial density (22) and evolving deterministically after that, denoting the associated empirical measure for $h \in \mathcal{C}^\infty(\mathcal{D},\mathbb{R})$

$$\pi_t^\varepsilon(h) = \frac{1}{\mu_\varepsilon}\sum_{i=1}^{\mathcal{N}} h(Z_i^\varepsilon(t)), \tag{64}$$

we know thanks to (33) that

$$\mathbb{E}_\varepsilon\left[\pi_t^\varepsilon(h)\right] = \int_{\mathcal{D}} F_1^\varepsilon(t,z)h(z)\mathrm{d}z, \tag{65}$$

so that for every $\delta > 0$, by Bienaymé-Chebishev inequality and Lanford theorem we have the following law of large numbers for the empirical measure

$$\mathbb{P}\left[\left|\pi_t^\varepsilon(h) - \int_{\mathcal{D}} f(t,z)h(z)\mathrm{d}z\right| > \delta\right] \xrightarrow[\mu_\varepsilon\to\infty]{} 0. \tag{66}$$

Similarly, with a similar theorem involving the second correlation function to control the variance of the empirical measure we can get the following central limit theorem

$$\zeta_t^\varepsilon(h) = \sqrt{\mu_\varepsilon}\left(\pi_t^\varepsilon(h) - \int_{\mathcal{D}} F_1^\varepsilon(t,z)h(z)\mathrm{d}z\right) \xrightarrow[\mu_\varepsilon\to\infty]{law} \mathcal{N}(0,1). \tag{67}$$

The following last probabilistic consequence was way more of a breakthrough when it was exposed in the 2015 article who owes it its name, by Thierry Bodineau, Isabelle Gallagher and Laure Saint-Raymond [2]. We don't expose the proof here since it does not depend at all on the formalism and stems from a theorem similar to Theorem 5.1.

Theorem 5.2 (Convergence in law to a Brownian motion)**.** *Let us fix a finite time $T > 0$ that will be sent to infinity by scaling thanks to the parameter α. Assuming that $\rho \in \mathcal{C}^0(\mathbb{T}^d)$, and denoting ρ^t the solution to the linear heat equation in $\mathbb{T}^d$*

$$\partial_t \rho^t(x) - \kappa_\beta \Delta_x \rho^t(x) = 0 \tag{68}$$

with initial condition ρ and diffusion coefficient

$$\kappa_\beta = \frac{1}{d} \int vQ(\cdot, M_\beta)^{-1} [vM_\beta(v)] \, \mathrm{d}v, \tag{69}$$

in the limit $\mu \to \infty$ and with the scaling $\alpha = \mu\varepsilon^{d-1} \to \infty$, but the latter convergence being slower than $\sqrt{\log\log\mu}$, we have

$$||F_1(\alpha\tau, x, v) - \rho^\tau(x) M_\beta(v)||_{\mathbb{L}^\infty} \longrightarrow 0. \tag{70}$$

Note that the speed of convergence allowed for α is a direct consequence of our main Theorem 5.1. Figure 4 illustrates this theorem: in this particular limit, the initial perturbation ρ - here chosen as an approximation on the torus of a gaussian - evolves slowly according to the heat equation until asymptotically reaching the uniform distribution. What is striking in this theorem is the fact that

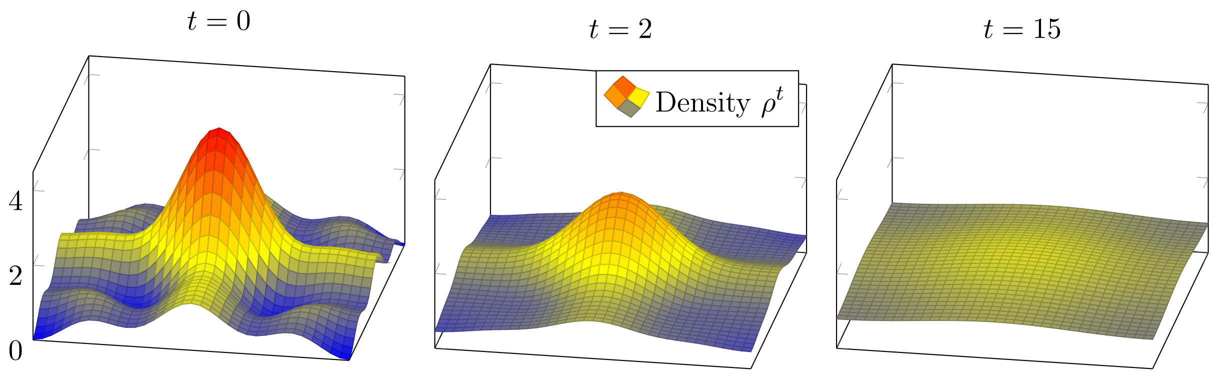


Figure 4: Evolution of the density ρ^t of the disturbed labelled particle (heat equation)

the evolution of the distribution of the tagged particle is similar to the evolution of the distribution of a Brownian motion, also given by the heat equation, so that the rescaled trajectory of this particle converges in law towards a Brownian motion!

6 Controlled pseudo-trajectories (outset of the proof of Theorem 5.1)

For simplicity, we denote $g(t, x, v) \doteq \varphi(t, x, v) M_\beta(v)$ our targeted density, and more generally

$$g_k(t, \underline{x}_n, \underline{v}_n) \doteq \varphi(t, x_1, v_1) M_\beta^{\otimes k}(\underline{v}_n). \tag{71}$$

This section is dedicated to the introduction of pseudo-trajectories, which are objects allowing to compare more easily both BBGKY and Boltzmann hierarchies by coupling them both. To maintain some bounds on the behavior of these pseudo-trajectories, bounds that will be foremost in the following sections, we will introduce the pruning of the trees of collisions associated to these trajectories. So as to estimate the error made when pruning, we will start stating some continuity estimates on the studied objects.

6.1 Continuity estimates

For writing simplicity, let us define the projection of $\mathcal{D}_p^\varepsilon$ on its space component only

$$\mathcal{X}_p^\varepsilon = \left\{ \underline{x}_p \in \left(\mathbb{T}^d\right)^p, \ \underline{z}_p \in \mathcal{D}_p^\varepsilon \right\}. \tag{72}$$

The first estimate is a bound on the correlation functions.

Proposition 6.1.1. *We have this first estimate, uniformly in time, over the correlation functions F_n for $n \geq 1$,*

$$\sup_{t\geq 0} F_n(t) \leq \|\rho\|_\infty M_\beta^{\otimes n} \mathbb{1}_{\mathcal{X}_n^\varepsilon}. \tag{73}$$

Proof. First of all, as for every $N \in \mathbb{N}$, W_N satisfies Liouville equation with initial conditions

$$W_N^0(\underline{z}_N) = \frac{\mu^N}{\mathcal{Z}} \rho(x_1) M_\beta^{\otimes N}(\underline{v}_N) \mathbb{1}_{\mathcal{X}_N^\varepsilon}(\underline{x}_N), \tag{74}$$

since transportation preserves the $\mathbb{L}^\infty$ norm, we still have that for every positive time $t \geq 0$

$$W_N(t, \underline{z}_N) \leq \frac{\mu^N}{\mathcal{Z}} \|\rho\|_\infty M_\beta^{\otimes N}(\underline{v}_N) \mathbb{1}_{\mathcal{X}_N^\varepsilon}(\underline{x}_N), \tag{75}$$

which integrated eventually yields the following control over the marginals of projected densities, for $s \leq N$

$$W_N^{(s)}(t, \underline{z}_s) \leq \frac{\mu^N}{\mathcal{Z}} \|\rho\|_\infty M_\beta^{\otimes s}(\underline{v}_s) \left(\mathbb{1}_{\mathcal{X}_N^\varepsilon}\right)^{(s)}(\underline{x}_s). \tag{76}$$

Hence, by definition of the correlation function (26), since all quantities are positive, we can write for any $\underline{z}_n \in \mathcal{D}^n$

$$\begin{aligned}
\sup_{t\geqslant 0} F_n(t, \underline{z}_n) &\leq \frac{1}{\mu^n} \sum_{p\geqslant 0} \frac{1}{p!} \sup_{t\geqslant 0} W_{n+p}^{(n)}(\underline{z}_n) && (77)\\
&\leq \frac{1}{\mathcal{Z}} \sum_{p\geqslant 0} \frac{\mu^p}{p!} \|\rho\|_\infty M_\beta^{\otimes n}(\underline{v}_n) \int \mathbb{1}_{\mathcal{X}_{n+p}^\varepsilon}(\underline{x}_{n+p}) \mathrm{d}x_{n+1} \dots \mathrm{d}x_{n+p} && (78)\\
&\leq \|\rho\|_\infty M_\beta^{\otimes n}(\underline{v}_n) \frac{\mathbb{1}_{\mathcal{X}_n^\varepsilon}(\underline{x}_n)}{\mathcal{Z}} \sum_{p\geqslant 0} \frac{\mu^p}{p!} \int \mathbb{1}_{\mathcal{X}_p^\varepsilon}(\underline{\tilde{x}}_p) \mathrm{d}\tilde{x}_1 \dots \mathrm{d}\tilde{x}_p && (79)\\
&= \|\rho\|_\infty M_\beta^{\otimes n}(\underline{v}_n) \mathbb{1}_{\mathcal{X}_n^\varepsilon}(\underline{x}_n). && (80)
\end{aligned}$$

□

Now, we will expose the morally very first step in the proof of the convergence of Theorem 5.1. Indeed, in the first place we need to show that at time $t = 0$, both BBGKY and limit distributions are close, if we want to show that they will remain close afterwards. In fact, both distributions satisfy the following kind of continuity estimate in ε.

Proposition 6.1.2 (Proximity of the initial distributions)**.** *In the scaling $\mu\varepsilon^{d-1} = \alpha$, and for a constant C_d depending only on the dimension, we have the following continuity estimate on the initial distributions before and at the limit*

$$\left|\left| \mathbb{1}_{\mathcal{X}_n^\varepsilon} g_n(0) - F_n(0) \right|\right|_{\mathbb{L}^\infty(\mathcal{D}^d)} \leq {C_d}^n \|\rho\|_{\mathbb{L}^\infty} \alpha\varepsilon. \tag{81}$$

Proof. By the definitions of the grand canonical (23) and canonical (32) partition functions we have for any $\underline{z}_n \in \mathcal{D}^n$, like in the proof of the previous Proposition 6.1.1

$$
\begin{aligned}
(\mathbb{1}_{\mathcal{X}_n^\varepsilon} g_n - F_n)(\underline{x}_n, \underline{v}_n) &= \rho(x_1) M_\beta^{\otimes n}(\underline{v}_n) \mathbb{1}_{\mathcal{X}_n^\varepsilon}(\underline{x}_n) - \frac{1}{\mathcal{Z}} \sum_{p\geq 0} \frac{\mu^p}{p!} \rho(x_1) M_\beta^{\otimes n}(\underline{v}_n) \left(\mathbb{1}_{\mathcal{X}_{n+p}^\varepsilon}\right)^{(n)}(\underline{x}_n) \\
&= \rho M_\beta^{\otimes n} \frac{1}{\mathcal{Z}} \sum_{p\geq 0} \frac{\mu^p}{p!} \left(\mathbb{1}_{\mathcal{X}_n^\varepsilon} \mathcal{Z}_p^c - \left(\mathbb{1}_{\mathcal{X}_{n+p}^\varepsilon}\right)^{(n)}\right), \qquad (82)
\end{aligned}
$$

and denoting $\mathbb{1}_{i\not\sim j} \doteq \mathbb{1}_{|x_i-x_j|>\varepsilon}$ and $\mathbb{1}_{i\sim j} \doteq \mathbb{1}_{|x_i-x_j|\leq\varepsilon}$, we can write

$$
\begin{aligned}
\mathbb{1}_{\mathcal{X}_n^\varepsilon} \mathcal{Z}_p^c - \left(\mathbb{1}_{\mathcal{X}_{n+p}^\varepsilon}\right)^{(n)} &= \mathbb{1}_{\mathcal{X}_n^\varepsilon} \int \prod_{n<i<j\leq n+p} \mathbb{1}_{i\not\sim j} \left(1 - \prod_{k\leq n<\ell\leqslant n+p} \mathbb{1}_{k\not\sim\ell}\right) \mathrm{d}x_{n+1} \dots \mathrm{d}x_{n+p} \\
&\leq \mathbb{1}_{\mathcal{X}_n^\varepsilon} \sum_{k\leq n<\ell\leqslant n+p} \int \mathbb{1}_{k\sim\ell} \prod_{n<i<j\leq n+p} \mathbb{1}_{i\not\sim j} \mathrm{d}x_{n+1} \dots \mathrm{d}x_{n+p} \\
&\leq \mathbb{1}_{\mathcal{X}_n^\varepsilon} \times np|\mathcal{B}_d|\varepsilon^d \int \prod_{n<i<j\leq n+p-1} \mathrm{d}x_{n+1} \dots \mathrm{d}x_{n+p-1} = \mathbb{1}_{\mathcal{X}_n^\varepsilon} np|\mathcal{B}_d|\varepsilon^d \mathcal{Z}_{p-1}^c,
\end{aligned}
$$

where we have integrated over x_{n+p} and used the following set theory inequality

$$
1 - \prod_i \mathbb{1}_{A_i} \leq \sum_i \mathbb{1}_{A_i^c}. \qquad (83)
$$

Hence, injecting this inequality in (82) we conclude the proof by writing

$$
\begin{aligned}
(\mathbb{1}_{\mathcal{X}_n^\varepsilon} g_n - F_n)(\underline{x}_n, \underline{v}_n) &\leq n\|\rho\|_{\mathbb{L}^\infty} M_\beta^{\otimes n} \frac{\mathbb{1}_{\mathcal{X}_n^\varepsilon}|\mathcal{B}_d|\varepsilon^d}{\mathcal{Z}} \sum_{p\geq 1} \frac{\mu^p}{(p-1)!} \mathcal{Z}_{p-1}^c \\
&\leq n\|\rho\|_{\mathbb{L}^\infty} \|M_\beta\|_{\mathbb{L}^\infty}^n |\mathcal{B}_d|\varepsilon^d \mu.
\end{aligned}
$$

□

We henceforth denote $H_k(\underline{v}_k)$ the total kinetic energy of the system

$$
H_k(\underline{v}_k) = \frac{1}{2} \sum_{i=1}^k |v_i|^2. \qquad (84)
$$

For $\lambda > 0$ holding the role of an inverse temperature and $k \in \mathbb{N}^*$, we consider the space $\mathcal{F}_{\varepsilon,k,\lambda}$ of measurable functions defined almost everywhere on the restricted domain $\mathcal{D}_k^\varepsilon$ such that

$$
\|f_k\|_{\varepsilon,k,\lambda} \doteq \operatorname*{supess}_{\underline{z}_k \in \mathcal{D}_k^\varepsilon} \Big| f_k(\underline{z}_k) \exp(\lambda H_k(\underline{v}_k)) \Big| < \infty, \qquad (85)
$$

and similarly in the limit $\varepsilon = 0$ the space $\mathcal{F}_{0,k,\lambda}$ of measurable functions defined almost everywhere on the whole domain $\mathcal{D}^k$ such that

$$
\|g_k\|_{0,k,\lambda} \doteq \operatorname*{supess}_{\underline{z}_k \in \mathcal{D}^k} \Big| g_k(\underline{z}_k) \exp(\lambda H_k(\underline{v}_k)) \Big| < \infty. \qquad (86)
$$

These functional spaces yield the right norms to state the following continuity estimate over the successive-collision operator modulus $|Q|_{s,s+n}$, defined as

$$
|Q|_{s,s+n}(t) \doteq \int_{T_n(t)} \Theta_s(t-t_1)|\mathcal{C}_s|\Theta_{s+1}(t_1-t_2)|\mathcal{C}_{s+1}| \dots \Theta_{s+n}(t_n) \mathrm{d}t_1 \dots \mathrm{d}t_n, \qquad (87)
$$

where the modulus of the collision operator $|\mathcal{C}_i|$ is simply given by (40) ; as well as over its limit version $|Q|_{s,s+n}^{\lim}$, formally given by the same formula taking limit operators $\Theta^{\lim}$ and $\mathcal{C}^{\lim}$.

Proposition 6.1.3. *There exists a constant C_d depending only on the dimension such that for all $s, n \in \mathbb{N}^*$, uniformly in time, if $f_{s+n} \in \mathcal{F}_{\varepsilon,s+n,\lambda}$, then*

$$|Q|_{s,s+n}(t) f_{s+n} \in \mathcal{F}_{\varepsilon,s,\lambda/2}$$

with

$$\left\| |Q|_{s,s+n}(t) f_{s+n} \right\|_{\varepsilon,s,\lambda/2} \le e^s \left(\frac{C_d t}{\lambda^{(d+1)/2}} \right)^n \|f_{s+n}\|_{\varepsilon,s+n,\lambda}. \tag{88}$$

The same result holds at the limit $\varepsilon = 0$ for the operator $|Q|^{\lim}_{s,s+n}$ and the corresponding functional space $\mathcal{F}_{0,k,\lambda}$ and its norm.

Let us observe that the correlation functions F_k will satisfy such continuity estimates since by Proposition 6.1.1 we have

$$\|F_k\|_{\varepsilon,k,\beta} \le \sup_{\underline{z}_k \in \mathcal{D}^\varepsilon_k} M_\beta^{\otimes k}(\underline{v}_k) \exp\big(\beta H_k(\underline{v}_k)\big) \|\rho\|_{\mathbb{L}^\infty} = \|\rho\|_{\mathbb{L}^\infty} \left(\frac{\beta}{2\pi}\right)^{\frac{kd}{2}}. \tag{89}$$

Similarly, at the limit $\varepsilon = 0$, the family $(M_\beta^{\otimes k}\rho)_k$ satisfying the limiting hierarchy (60) will also satisfy the same inequality, and thus such continuity estimates. Finally, this estimate justifies the convergence of the Duhamel infinite series (43) in $\mathcal{F}_{\varepsilon,s,\beta/2}$ for times $t \ge 0$ small enough.

Proof. First of all, let us remark that the transport operators preserve all the weighted norms $\|\cdot\|_{\varepsilon,s,\lambda}$ since the weight is only along the velocities.

Then, let us compute for $f_{j+1} \in \mathcal{F}_{\varepsilon,j+1,\lambda}$

$$\begin{aligned}
\Big|\Theta_j(-\tau)\mathcal{C}_j\Theta_{j+1}(\tau)f_{j+1}\Big| &= \left|\Theta_j(-\tau)\sum_{i=1}^j \int \omega\cdot(v_{j+1}-v_i)\Theta_{j+1}(\tau)f_{j+1}(\underline{z}_j, x_i+\varepsilon\omega, v_{j+1})\mathrm{d}\omega\mathrm{d}v_{j+1}\right| \\
&\le \sum_{i=1}^j \int (|v_{j+1}|+|v_i|)\|f_{j+1}\|_{\varepsilon,j+1,\lambda}\exp\left[-\lambda H_{j+1}(\underline{v}_{j+1})\right]\mathrm{d}\omega\mathrm{d}v_{j+1} \\
&\le |\mathbb{S}^{d-1}|\cdot\|f_{j+1}\|_{\varepsilon,j+1,\lambda}\sum_{i=1}^j\int(|v_{j+1}|+|v_i|)\exp\left[-\frac{\lambda}{2}\sum_{k=1}^{j+1}|v_k|^2\right]\mathrm{d}v_{j+1}.
\end{aligned}$$

The remaining integral may be written explicitly up to constants depending only on the dimension d,

$$\begin{aligned}
\int (|v_{j+1}|+|v_i|)\exp\left[-\frac{\lambda}{2}\sum_{k=1}^{j+1}|v_k|^2)\right]\mathrm{d}v_{j+1} &= C_d\int(r+|v_i|)\exp\left[-\frac{\lambda}{2}\sum_{k=1}^{j}|v_k|^2\right]r^{d-1}e^{-\frac{\lambda}{2}r^2}\mathrm{d}r \\
&= C_d\exp\Big[-\lambda H_k(\underline{v}_k)\Big]\left(c_d\sqrt{\lambda^{-(d+1)}}+|v_i|\tilde{c}_d\sqrt{\lambda^{-d}}\right).
\end{aligned} \tag{90}$$

This way, putting it altogether by summing (90) in i and accepting to lose a $\lambda/2n$ in the considered norm so as to absorb the factors $|v_i|$, we get that

$$\begin{aligned}
&\Big\|\Theta_{s+n-1}(-t_n)|\mathcal{C}_{s+n}|\Theta_{s+n}(t_k)f_{s+n}\Big\|_{\varepsilon,s+k-1,\lambda-\lambda/2n} \\
&\quad\le \widetilde{C}_d\left((s+n-1)\sqrt{\lambda^{-(d+1)}}+\sqrt{\lambda^{-d}}\sum_{i=1}^{s+n-1}|v_i|\right)\exp\left[-\frac{\lambda}{4n}\sum_{k=1}^{s+n-1}|v_k|^2\right],
\end{aligned}$$

but using Cauchy-Schwarz inequality and the fact that $\left[x \geq 0 \Rightarrow xe^{-x} \leq e^{-1}\right]$ we have

$$\left(\sum_{i=1}^{s+n-1} |v_i|\right) \exp\left[-\frac{\lambda}{4n}\sum_{k=1}^{s+n-1}|v_k|^2\right] \leq \left(\frac{(s+n-1)2n}{\lambda}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{s+n-1}|v_i|^2\frac{\lambda}{2n}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{4n}\sum_{k=1}^{s+n-1}|v_k|^2}$$
$$\leq \left(\frac{(s+n)2n}{e\lambda}\right)^{\frac{1}{2}} \leq \sqrt{\frac{2}{e\lambda}}(s+n).$$

Iterating this calculus, losing a $\lambda/2n$ in the considered norm and gaining a factor $\widehat{C}_d(s+n)\sqrt{\lambda^{-(d+1)}}$ at each iteration we obtain

$$\left\| |Q|_{s,s+n}(t) f_{s+n}\right\|_{\varepsilon,s,\lambda/2} \leq \left(\widehat{C}_d(s+n)\sqrt{\lambda^{-(d+1)}}\right)^n \int_{T_n(t)} \mathrm{d}t_1 \cdots \mathrm{d}t_n \|f_{s+n}\|_{\varepsilon,s+n,\lambda}$$
$$\leq \widehat{C}_d^n\,(s+n)^n\sqrt{\lambda^{-n(d+1)}}\;\frac{t^n}{n!}\;\|f_{s+n}\|_{\varepsilon,s+n,\lambda}$$
$$\leq \widehat{C}_d^n e^{s+n}\sqrt{\lambda^{-n(d+1)}}\;t^n\;\|f_{s+n}\|_{\varepsilon,s+n,\lambda},$$

where the factor $n!$ comes from the imposed order of collision times in $T_n(t)$, and allows to control the term $(s+n)^n$, concluding the proof.

□

6.2 Pseudo-trajectories

From now on, we will restrain ourselves to the framework of Theorem 5.1, studying mainly the first correlation function.

First of all, we will provide a different way to write the series satisfied by our BBGKY and Boltzmann hierarchies, so as to be able to compare them more easily. This new formulation is based on the idea of *coupled pseudo-trajectories*, that was already introduced by Lanford in his first article. These trajectories will be easily comparable because the velocities and deflection angles they will be built with, appearing as integration variables, will be the same for both of them. Let us recall the general form of the terms in the expansion series (43)

$$Q_{1,1+k}(t)F_{1+k}(0) = \int_{T_k(t)} \Theta_1(t-t_1)\mathcal{C}_1\Theta_2(t_1-t_2)\mathcal{C}_2 \ldots \Theta_{1+k}(t_k)F_{1+k}(0)\mathrm{d}t_1 \ldots \mathrm{d}t_k,$$

where the collision operators C_i are integrals over a velocity v_{i+1} and a deflection angle ω_{i+1} (37). Hence, at fixed times $0 \leq t_k \leq \cdots \leq t_1 \leq t$, velocities $v_2, \ldots, v_{1+k}$ and deflection angles $\omega_2, \ldots, \omega_{1+k}$, we will give an interpretation of the integrand in terms of pseudo-trajectories (which are not really physical trajectories, but rather abstract objects resembling to trajectories). *These trajectories are constructed backwards,* starting from time t and adding pseudo-particles at decreasing times until reaching $t = 0$. Indeed, to study the system at time t, we want to avoid that colliding particles had already collided before, so as to control more easily where they come from and thus their initial distribution, so that we have to study what was happening in the past, backward in time. For this reason, we will mostly consider pre-collisional velocities, denoted $(v^*, {v_c}^*)$ (see Section 2.1, (8)).

Each collision operator C_i is seen as a pseudo-collision happening at time t_i with parameters v_{i+1}, ω_{i+1}. Indeed, one can see in the formulation (39) that in the integral defining C_i we add a new particle with position $x_i + \varepsilon\omega_{i+1}$ and velocity v_{i+1} or ${v_{i+1}}'$ – depending if the velocity is pre- or post-collisional. Besides adding a new particle, we also change or not the velocity v_i into ${v_i}'$ – depending on the same condition. Given the state of a single pseudo-particle $(x_1(t), v_1(t))$ at time t, we then imagine that its pseudo-trajectory $(x_1(u), v_1(u))$ was following the free flow in the interval $[t_1, t]$, so that $x_1(t_1) = x_1(t) - (t - t_1)v_1(t)$. At time t_1, another pseudo-particle had collided with our first particle with deflection angle ω_2 and velocity v_2 – as a pre- or post-collisional velocity according to the sign of $\omega_2 \cdot (v_2 - v_1)$ – so that its pseudo-trajectory satisfies

$$z_2(t_1^-) = \begin{cases} (\ x_1(t_1), v_1, x_1(t_1) + \varepsilon\omega_2, v_2\ ) & \text{if } \omega_2 \cdot (v_2 - v_1) < 0 \\ (\ x_1(t_1), {v_1}^*, x_1(t_1) + \varepsilon\omega_2, {v_2}^*) & \text{otherwise.} \end{cases} \tag{91}$$

Let us insist on the fact that when v_2 is precollisional, the first particle *does not change velocity*, the trajectory does not present a real collision, we only add a new pseudo-particle. In this construction, by conservation of the energy, the kinetic energy is always given by

$$H_k(\underline{z}_k) = \sum_{i=1}^{k} |v_i|^2. \tag{92}$$

Before this collision at time t_1, or this ghost collision in the pre-collisional case, both particles were following the two-particle free-flow with specular reflection, having possibly collided before - which situation is usually called *recollision*, looking time backwards. Adding a third pseudo-particle at time t_2 with parameters v_3, ω_3, we have to choose what particle it had collided among the two first ones. We denote $m_2 \in \{1, 2\}$ the tag of this particle, chosen among available particles. Hence, iterating this construction, we eventually obtain the *list of tags* $\underline{m} = (m_1, \ldots, m_k)$ of the particles that have collided, and $k+1$ pseudo-trajectories $(x_i(u), v_i(u))$ defined for $u \in [0, t_{i-1}]$. Let us remark that these pseudo-trajectories very strongly depend on the diameter ε. In particular, some recollisions may happen or not depending on the diameter of the spheres. These trajectories hence define a tree of collisions, branching at each collision time. Figure 5 illustrates such a tree construction, for 3 collision times, a recollision and an added particle whose velocity v_4 is precollisional, so that the associated collision at t_3 is a ghost collision that *do not deflect* the particle tagged $m_4 = 2$.

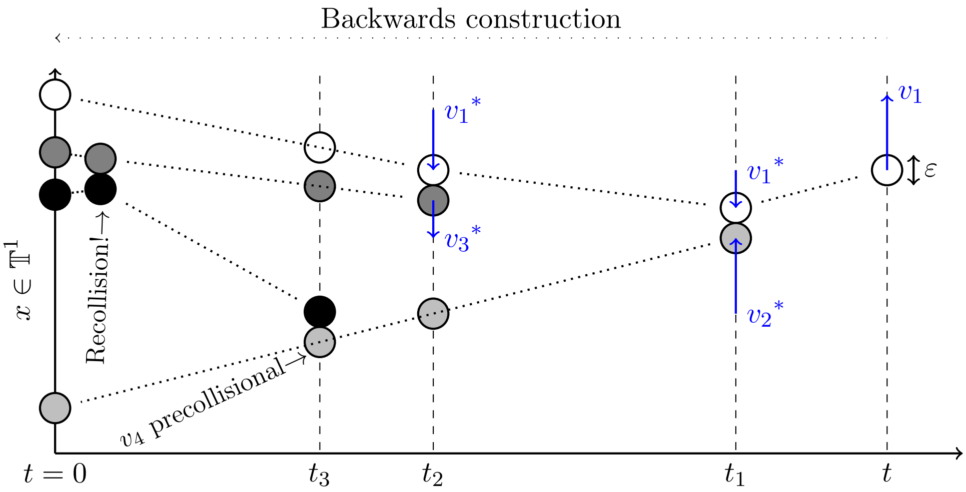


Figure 5: Backwards construction of a tree of collisions for pseudo-trajectories

Similarly, we define the pseudo-trajectories for the limiting model corresponding formally to the case $\varepsilon = 0$, coupled to the first ones if chosing the same collision parameters. In the limiting model, recollisions are of measure zero since particles are points. The whole point of Section 7 is to show that one can consider only BBGKY pseudo-trajectories that *do not recollide*, which would hence be close to the Boltzmann pseudo-trajectories.

6.3 Tree pruning

Now, so as to work with collision trees of controlled size (which will provide reasonable bounds in the final estimates of Section 8), we will simply consider truncated series instead of our Duhamel series expansions (43). More precisely, for a fixed time $t > 1$ we will study how our functionals behave on the interval $[0, t]$, and start cutting this interval into $K \in \mathbb{N}^*$ little pieces of size $h = T/K$:

$$[0, t] = [0, h] \cup [h, 2h] \cup \cdots \cup [(K-1)h, Kh].$$

We will now morally forbid more than $A \geq 2$ collisions per particle to happen in each small interval: at the k-th time quantum of length h, we want that at most A^k particles may have collided, as if we were pruning the collision tree when it becomes more than exponentially big. To get an explicit formulation of this condition on the series expansion, we will write the series (43) between $t - h$ and t, cut after A collisions, then do it again between $t - 2h$ and $t - h$ after A^2 collisions, and eventually iterate this calculus:

$$\begin{aligned}
F_1(t) &= \sum_{j_1=0}^{A-1} \alpha^{j_1} Q_{1,1+j_1}(h) F_{1+j_1}(t-h) + \sum_{s=A}^{\infty} \alpha^s Q_{1,1+s}(h) F_{1+s}(t-h) \\
&= \sum_{j_1=0}^{A-1} \sum_{j_2=0}^{A^2-1} \alpha^{j_1+j_2} Q_{1,1+j_1}(h) Q_{1+j_1,1+j_1+j_2}(h) F_{1+j_1+j_2}(t-2h) + \sum_{s=A}^{\infty} \alpha^s Q_{1,1+s} F_{1+s}(t-h) \\
&\quad + \sum_{j_1=0}^{A-1} \alpha^{j_1} Q_{1,1+j_1}(h) \left[\sum_{s=A^2}^{\infty} \alpha^s Q_{1+j_1,1+j_1+s}(h) F_{1+j_1+s}(t-2h) \right] \\
&= \sum_{j_1=0}^{A-1} \cdots \sum_{j_K=0}^{A^K-1} \alpha^{J_K-1} Q_{1,J_1}(h) Q_{J_1,J_2}(h) \dots Q_{J_{K-1},J_K}(h) F_{J_K}(0) \\
&\quad + \sum_{k=1}^{K} \sum_{j_1=0}^{A-1} \cdots \sum_{j_{k-1}=0}^{A^{k-1}-1} \alpha^{J_{k-1}-1} Q_{1,J_1}(h) \dots Q_{J_{k-2},J_{k-1}}(h) \sum_{s=A^k}^{\infty} \alpha^s Q_{J_{k-1},J_{k-1}+s}(h) F_{J_{k-1}+s}(t-kh)
\end{aligned}$$

denoting

$$J_K \doteq 1 + j_1 + \cdots + j_K.$$

We hence introduce the truncated correlation function

$$F_1^{[K]}(t) \doteq \sum_{j_1=0}^{A-1} \cdots \sum_{j_K=0}^{A^K-1} \alpha^{J_K-1} Q_{1,J_1}(h) Q_{J_1,J_2}(h) \dots Q_{J_{K-1},J_K}(h) F_{J_K}(0) \tag{93}$$

and its remainder, which corresponds to the pruned-out trajectories,

$$R^{[K]}(t) \doteq \sum_{k=1}^{K} \sum_{j_1=0}^{A-1} \cdots \sum_{j_{k-1}=0}^{A^{k-1}-1} \alpha^{J_{k-1}-1} Q_{1,J_1}(h) \dots Q_{J_{k-2},J_{k-1}}(h) \sum_{s=A^k}^{\infty} \alpha^s Q_{J_{k-1},J_{k-1}+s}(h) F_{J_{k-1}+s}(t-kh). \tag{94}$$

We define similarly the limit equivalents of these two functionals, denoted $g^{[K]}$ and $R^{[K]}_{\text{lim}}$, by merely replacing the successive-collision operators $Q_{i,j}$ by their limit version $Q^{\text{lim}}_{i,j}$.

We will study the truncated function in the following section, but first we have to justify that the remainder may be neglicted. This is the point of the following proposition, using the continuity estimates of previous section.

Proposition 6.3.1 (Estimates of the pruned-out terms)**.** *With previous notation, there are three constants $c_{d,\beta}, C_{d,\beta,A} \in \mathbb{R}^+$ and $\eta_0 < 1$ such that for any $t > 1$, taking*

$$h = \frac{c_{d,\beta}\eta_0}{\alpha^{A/(A-1)}t^{1/(A-1)}}, \tag{95}$$

one has

$$\left|\left|R^{[K]}(t)\right|\right|_{\mathbb{L}^\infty(\mathcal{D}^d)} + \left|\left|R^{[K]}_{\lim}(t)\right|\right|_{\mathbb{L}^\infty(\mathcal{D}^d)} \leq C_{d,\beta,A}\|\rho\|_{\mathbb{L}^\infty}\eta_0^A. \tag{96}$$

Proof. Let us study the general term of the series, which satisfies

$$\begin{aligned}&\left|\left|Q_{1,J_1}(h)\cdots Q_{J_{k-2},J_{k-1}}(h)\sum_{s=A^k}^{\infty}\alpha^s Q_{J_{k-1},J_{k-1}+s}(h)F_{J_{k-1}+s}(t-kh)\right|\right|_{\mathbb{L}^\infty(\mathcal{D}^d)}\\ &\qquad\leq \left|\left|\,|Q|_{1,J_{k-1}}((k-1)h)\sum_{s=A^k}^{\infty}\alpha^s|Q|_{J_{k-1},J_{k-1}+s}(h)F_{J_{k-1}+s}(t-kh)\right|\right|_{\mathbb{L}^\infty(\mathcal{D}^d)},\end{aligned}$$

since the right term is exactly the left one but where we have lifted some conditions on the order of the integrating variables, which corresponds to removing indicator functions, and lowers the norm by positivity when considering the modulus of the operators.

As $F_{J_k+s} \in \mathcal{F}_{\varepsilon,J_k+s,\beta}$ by Proposition 6.1.1, the continuity estimate on the successive-collision operators $Q_{i,i+j}$ given in Proposition 6.1.3, along with the following calculus, asserts that

$$\sum_{s=A^k}^{\infty}\alpha^s|Q|_{J_{k-1},J_{k-1}+s}(h)F_{J_{k-1}+s}(t-kh) \in \mathcal{F}_{\varepsilon,J_{k-1},\beta/2}$$

and then that, since the infinity norm is bounded by all the weighted norms,

$$\begin{aligned}&\left|\left|\,|Q|_{1,J_{k-1}}(h)\sum_{s=A^k}^{\infty}\alpha^s|Q|_{J_{k-1},J_{k-1}+s}(h)F_{J_k+s}(t-kh)\right|\right|_{\mathbb{L}^\infty(\mathcal{D}^d)}\\ &\leq \left(\frac{C_d(k-1)h}{\beta^{(d+1)/2}}\right)^{J_{k-1}-1}\left|\left|\sum_{s=A^k}^{\infty}\alpha^s|Q|_{J_{k-1},J_{k-1}+s}(h)F_{J_{k-1}+s}(t-kh)\right|\right|_{\varepsilon,J_{k-1},\beta/2}\\ &\leq \left(\frac{C_d(k-1)h}{\beta^{(d+1)/2}}\right)^{J_{k-1}-1}\sum_{s=A^k}^{\infty}\left(\frac{C_d\alpha h}{\beta^{(d+1)/2}}\right)^s\left|\left|F_{J_{k-1}+s}(t-kh)\right|\right|_{\varepsilon,J_{k-1}+s,\beta}\\ &\leq \left(\frac{C_d(k-1)h}{\beta^{(d+1)/2}}\right)^{J_{k-1}-1}\sum_{s=A^k}^{\infty}\left(\frac{C_d\alpha h}{\beta^{(d+1)/2}}\right)^s\|\rho\|_{\mathbb{L}^\infty}\left(\frac{\beta}{2\pi}\right)^{\frac{(J_{k-1}+s)d}{2}}\\ &\leq (C_dt)^{J_{k-1}-1}\sqrt{\beta}^{d+1-J_{k-1}}\sum_{s=A^k}^{\infty}\left(\frac{C_d\alpha h}{\sqrt{\beta}}\right)^s\|\rho\|_{\mathbb{L}^\infty}\left(\frac{1}{2\pi}\right)^{\frac{(J_{k-1}+s)d}{2}},\end{aligned}$$

recalling that $(k-1)h < t$, and where the penultimate inequality is a consequence of the estimates provided by Proposition 6.1.1 and (89). Let us remark that the terms in e^s in the continuity estimates have been hidden in the constant C_d. Hence, hiding the 2π factor in another constant $\widetilde{C}_d$, under the following additional assumption on h

$$\frac{\widetilde{C}_d\alpha h}{\sqrt{\beta}} < \frac{1}{2}, \tag{97}$$

we find

$$\begin{aligned}
&\left\|\left|Q\right|_{1,J_{k-1}}(h)\sum_{s=A^k}^{\infty}\alpha^s|Q|_{J_{k-1},J_{k-1}+s}(h)F_{J_{k-1}+s}(t-kh)\right\|_{\mathbb{L}^\infty(\mathcal{D}^d)}\\
&\qquad\le \|\rho\|_{\mathbb{L}^\infty}\left(\widetilde{C}_d t\right)^{J_{k-1}-1}\sqrt{\beta}^{d+1-J_{k-1}}\times 2\left(\frac{\widetilde{C}_d\alpha h}{\sqrt{\beta}}\right)^{A^k}\\
&\qquad\le \|\rho\|_{\mathbb{L}^\infty}\frac{\widehat{C}_d^{J_{k-1}-1+A^k}}{\sqrt{\beta}^{J_{k-1}+A^k-d-1}}t^{J_{k-1}-1}\left(\alpha h\right)^{A^k}.
\end{aligned}$$

Let us remark that $J_{k-1}\le 1+A+\cdots+A^{k-1}=\dfrac{A^k}{A-1}$ so that, since by hypothesis $\alpha t\ge 1$ and up to artificially chose a bigger constant,

$$\begin{aligned}
&\alpha^{J_{k-1}-1}\left\|\left|Q\right|_{1,J_{k-1}}(h)\sum_{s=A^k}^{\infty}\alpha^s|Q|_{J_{k-1},J_{k-1}+s}(h)F_{J_{k-1}+s}(t-kh)\right\|_{\mathbb{L}^\infty(\mathcal{D}^d)}\\
&\qquad\le \|\rho\|_{\mathbb{L}^\infty}\beta^{\frac{d}{2}}\left(\frac{\widehat{C}_d}{\sqrt{\beta}}\right)^{\frac{A^{k+1}}{A-1}}\left(\alpha t\right)^{\frac{A^k}{A-1}}\left(\alpha h\right)^{A^k}.
\end{aligned}$$

Hence, as soon as

$$h\le\frac{\eta_0}{(\widehat{C}_d/\sqrt{\beta})^{A/(A-1)}\alpha^{A/(A-1)}t^{1/(A-1)}},$$

which is compatible with condition (97) for $\eta_0>0$ small enough, we get

$$\alpha^{J_{k-1}-1}\left\|\left|Q\right|_{1,J_{k-1}}(h)\sum_{s=A^k}^{\infty}\alpha^s|Q|_{J_{k-1},J_{k-1}+s}(h)F_{J_{k-1}+s}(t-kh)\right\|_{\mathbb{L}^\infty(\mathcal{D}^d)}\le\|\rho\|_{\mathbb{L}^\infty}\beta^{\frac{d}{2}}\eta_0^{A^k}.$$

Henceforth, we may compute the estimate on the remainder (94)

$$\begin{aligned}
\left\|R^{[K]}(t)\right\|_{\mathbb{L}^\infty}&\le\|\rho\|_{\mathbb{L}^\infty}\beta^{\frac{d}{2}}\sum_{k=1}^{K}A^{k(k+1)/2}\eta_0^{A^k}\\
&\le\|\rho\|_{\mathbb{L}^\infty}\beta^{\frac{d}{2}}\sum_{k=1}^{K}\exp\left(\frac{k(k+1)}{2}\log A+A^k\log\eta_0\right).
\end{aligned}$$

Since $A>2$ and assuming a simple bound like $\eta_0<1/2$, the term in the exponential will be smaller than $C_AA^k\log\eta_0$ for a constant C_A depending on A. Hence, with the same bound on η_0, we have

$$\begin{aligned}
\left\|R^{[K]}(t)\right\|_{\mathbb{L}^\infty}&\le\|\rho\|_{\mathbb{L}^\infty}\beta^{\frac{d}{2}}e^{C_A}\sum_{k=1}^{K}\eta_0^{kA}\\
&\le\|\rho\|_{\mathbb{L}^\infty}\beta^{\frac{d}{2}}e^{C_A}\times 2\eta_0^{A},
\end{aligned}$$

which concludes the proof (calculus is the same for the limit remainders).

□

6.4 Reformulation in terms of pseudo-trajectories

With the same notation, and denoting $\underline{j} \doteq (j_1, \ldots, j_K)$, in our new truncated expansion series $F_1^{[K]}$ (93), the general term is now of the form

$$
\begin{aligned}
f_{\underline{j}}^{[K]}(t) &\doteq Q_{1,J_1}(h) Q_{J_1,J_2}(h) \ldots Q_{J_{K-1},J_K}(h) F_{J_K}(0) \\
&= \int_{T_{\underline{j}}^h(t)} \Theta_1(t-t_1)\mathcal{C}_1 \Theta_2(t_1-t_2)\mathcal{C}_2 \ldots \Theta_{J_K}(t_{J_K-1}) F_{J_K}(0) \, \mathrm{d}\underline{t}_{J_K-1},
\end{aligned} \tag{98}
$$

with the following additional condition on the successive collision times (61)

$$
\underline{t}_{J_K-1} \in T_{\underline{j}}^h(t) \doteq \Big\{ (t_1, \ldots, t_{J_K-1}) \in T_{J_K-1}(t) \ , \ \{t_{J_k}, \ldots, t_{J_{k+1}-1}\} \subset [t-kh, t-(k-1)h] \Big\}, \tag{99}
$$

and the convention $t_0 = t$ and $t_{J_K} = 0$. Now, denoting

$$
\mathcal{M}_{\underline{j}} \doteq \{ \underline{m}_{J_K-1} = (m_1, \ldots, m_{J_K-1}) \ , \ 1 \leq m_i \leq i \}, \tag{100}
$$

we can write these functions in terms of pseudo trajectories in the following way

$$
f_{\underline{j}}^{[K]}(t, z_1) = \sum_{\underline{m} \in \mathcal{M}_{\underline{j}}} \int_{T_{\underline{j}}^h(t)} \mathrm{d}\underline{t}_{J_K-1} \int_{(\mathbb{S}^{d-1} \times \mathbb{R}^d)^{J_K-1}} \mathrm{d}\underline{\omega} \mathrm{d}\underline{v} \prod_{i=1}^{J_K-1} \Big[(v_{i+1} - v_{m_i}) \cdot \omega_{i+1} \Big] F_{J_K}\big(0, \underline{z}_{J_k}(0)\big), \tag{101}
$$

where $\underline{z}_{J_k}(0)$ are the pseudo-trajectories given by $z_1(t)$, $\underline{m} = \underline{m}_{J_K-1}$, $\underline{v} = \underline{v}_{J_K-1}$ and $\underline{\omega} = \underline{\omega}_{J_K-1}$ (see Section 6.2). Now for the limit case this formulation is very similar: the only difference happens in the initial distribution of the pseudo-trajectories:

$$
\overline{g}_{\underline{j}}^{[K]}(t, z_1) = \sum_{\underline{m} \in \mathcal{M}_{\underline{j}}} \int_{T_{\underline{j}}^h(t)} \mathrm{d}\underline{t}_{J_K-1} \int_{(\mathbb{S}^{d-1} \times \mathbb{R}^d)^{J_K-1}} \mathrm{d}\underline{\omega} \mathrm{d}\underline{v} \prod_{i=1}^{J_K-1} \Big[(v_{i+1} - v_{m_i}) \cdot \omega_{i+1} \Big] g_{J_K}\big(0, \underline{z}_{J_k}^{\lim}(0)\big). \tag{102}
$$

For this reason, next section will introduce tools to compare the pseudo-trajectories before and at the limit.

7 Non-recollisioning paths (continuation of the proof)

Indeed in this section, we will provide tools to quantify how close are the limiting pseudo-trajectories and the BBGKY ones with positive diameters $\varepsilon > 0$. Since the particles in the limit trajectories are points, they do not involve recollisions. Hence, to know how close both trajectories may be, we have to control recollisions for the non-point ones.

7.1 Particle adjunction at bounded energy for non-pathological parameters

Denoting for $x, y \in \mathbb{R}^d$

$$
d(x, y) = \min_{k \in \mathbb{Z}^d} |x - y - k| \tag{103}
$$

the distance on the torus, we introduce the set of pseudo-trajectories whose past spatial trajectories are independent and do not get closer one to another than a constant $\varepsilon_0 > 0$

$$
I_k(\varepsilon_0) = \Big\{ \underline{z}_k \in \mathcal{D}^k \ , \ \forall \tau \in [0, t], \forall i \neq j, d(x_i - \tau v_i, x_j - \tau v_j) \geq \varepsilon_0 \Big\}. \tag{104}
$$

In these trajectories, as particles do not collide, they all follow a free transport flow.

In order to obtain further results, we will have to restrain ourselves to bounded energies. We hence introduce for an energy $p^2 > 0$ the ball

$$\mathcal{B}_p \doteq \{v \in \mathbb{R}^d \ , \ |v|^2 \leq p^2\}. \tag{105}$$

The error associated to this restriction will be controlled thanks to the weighted norms in section 8.1, and the energy will be sent to infinity with the right convergence speed. Furthermore, we will introduce the following parameters that we will also eventually scale in the end of Section 8

$$A^{K+1}\varepsilon \leq a \ll \varepsilon_0 \ll \eta\delta \ll \min(\delta p, 1). \tag{106}$$

The parameters a and ε_0 stand for space distances between particles; the introduction of η, like p, is due to a truncation in velocities at a different scale, and δ stands for a time delay. These parameters are all involved so as to state the following proposition, whose proof may be found in [5, Chapter 12]. Let us observe that in the first version of [5], this proposition was stated in a wrong way that has been corrected since, but remains in [2].

Proposition 7.1.1 (Non pathological particle adjunction)**.** *Given a finale limit configuration* $\underline{z}_k^{\lim} = (\underline{x}_k^{\lim}, \underline{v}_k) \in I_k(\varepsilon_0)$ *and a collisioning tag* $m_k \leq k$*, there is a subset* $\Pi_k^{m_k}(\underline{z}_k^{\lim}) \subset \mathbb{S}^{d-1} \times \mathcal{B}_p$ *of pathological collision parameters, with small measure*

$$|\Pi_k^{m_k}(\underline{z}_k^{\lim})| \leq Ck \left(\eta^d + p^d \left(\frac{a}{\varepsilon_0}\right)^{\frac{d-1}{2}} + p^{\frac{d+1}{2}} \left(\frac{\varepsilon_0}{\delta}\right)^{\frac{d-1}{2}} \right), \tag{107}$$

such that other parameters provide a stability condition when adding a particle to a configuration $\underline{z}_k$ *close to* $\underline{z}_k^{\lim}$*, in the following sense.*

For a spatial configuration $\underline{x}_k$ *such that* $|x_k - x_k^{\lim}| \leq a$*, and non-pathological collision parameters* $(\omega_{k+1}, v_{k+1}) \in \mathbb{S}^{d-1} \times \mathcal{B}_p \setminus \Pi_k^{m_k}(\underline{z}_k^{\lim})$*, adding a new pseudo-particle at position* $x_{k+1} = x_{m_k} + \varepsilon\omega_{k+1}$ *to* $\underline{z}_k = (\underline{x}_k, \underline{v}_k)$*, then*

$$\forall \tau \in]0, t], \forall i \neq j, d(x_i - \tau\tilde{v}_i, x_j - \tau\tilde{v}_j) > \varepsilon, \tag{108}$$

where $\tilde{v_i} \in \{v_i, {v_i}^*\}$ *is the updated velocity in the case of a post-collisional configuration. Moreover, after a time* δ*, the particles will still evolve indepently*

$$\forall \tau \in [\delta, t], \begin{cases} (\underline{x}_{k+1} - \tau\underline{\tilde{v}}_{k+1}, \underline{\tilde{v}}_{k+1}) \in I_{k+1}(\varepsilon_0/2) \\ (\underline{x}_{k+1}^{\lim} - \tau\underline{\tilde{v}}_{k+1}, \underline{\tilde{v}}_{k+1}) \in I_{k+1}(\varepsilon_0). \end{cases} \tag{109}$$

7.2 Inductive construction of non-recollisioning paths

Since in previous proposition the system needs a delay δ before going back to a good state, we consider the set of times satisfying the delay condition

$$T_{\underline{j}}^{h,\delta}(t) = \left\{ \underline{t} \in T_{\underline{j}}^h(t) \ , \ t_{i-1} - t_i > \delta \right\}. \tag{110}$$

Hence, if one inductively avoids the pathological collision parameters and lets the system stabilize between the collisions, then both coupled pseudo-trajectories will be close one to another. With the same parameters (106) as in Proposition 7.1.1, we thus have the following proposition.

Proposition 7.2.1. *Let us fix* $\underline{j} = (j_1, \dots, j_K)$*,* $\underline{m} = (m_1, \dots, m_{J_K-1})$ *and* $\underline{t} \in T_{\underline{j},\delta}^h(t)$*, for a given* $z_1(t)$*, and let us denote* $\underline{z}_i$ *and* $\underline{z}_i^{\lim}$ *the pseudo-trajectories associated to this same finale state* $z_1(t)$ *and to collision parameters inductively chosen such that*

$$(\omega_{i+1}, v_{i+1}) \in \mathbb{S}^{d-1} \times \mathcal{B}_p \setminus \Pi_i^{m_i}(\underline{z}_i^{\lim}(t_i)) \text{ and } \sum_{j=1}^{i+1} v_j^2 < p^2. \tag{111}$$

Then, for ε small enough, the velocities of both pseudo-trajectories coincide, as well as the positions of the first tagged particle $x_1(\tau)$ and $x_1^{\lim}(\tau)$ for all $\tau \in [0,t]$. Additionally, we have the following spatial proximity

$$\forall i \leq J_K - 1,\ \forall \ell \leq i+1,\ |x_\ell(t_{i+1}) - x_\ell^{\lim}(t_{i+1})| \leq \varepsilon i. \tag{112}$$

For the following, we will introduce the *set of pathological sequences of collision parameters*

$$\Pi(z_1, \underline{j}, \underline{t}, \underline{m}) \doteq \Bigg\{ (\omega_i, v_i)_{2\leq i \leq J_K}\ ,\ \sum_{j=1}^{J_K} v_j^2 < p^2 \tag{113}$$

$$\text{and } \exists i_0 \leq J_K - 1 :\ \underline{z}_{i_0}^{\lim}(t_{i_0}) \in I_{i_0}(\varepsilon_0) \text{ and } (\omega_{i_0+1}, v_{i_0+1}) \in \Pi_{i_0}^{m_{i_0}}(\underline{z}_{i_0}^{\lim}(t_{i_0})) \Bigg\}.$$

Proof. Let us prove this proposition by induction on $1 \leq i \leq J_K$, with the recursion hypothesis

$$\underline{z}_i^{\lim}(t_i) \in I_i(\varepsilon_0) \text{ and } \forall \ell \leq i, \Big[v_\ell(t_i) = v_\ell^{\lim}(t_i) \text{ and } |x_\ell(t_i) - x_\ell^{\lim}(t_i)| \leq \varepsilon(i-1) \Big], \tag{114}$$

which is true by construction for $i = 1$. If then the recursion hypothesis (114) is true up to an index $i \leq J_K - 1$, let us prove that it still holds for the index $i+1$. Both pre- and post-situations behave the same, noting merely that the constraint

$$\sum_{j=1}^{i+1} v_j^2 < p^2$$

on the velocities also implies the same constraint for their pre-collisional counterparts, by conservation of momentum, so that all velocities will stay in $\mathcal{B}_p$.

Let us then add a $(i+1)$-th particle at time t_i with collision parameters $(\omega_{i+1}, v_{i+1}) \in \mathbb{S}^{d-1} \times \mathcal{B}_p \setminus \Pi_i^{m_i}(\underline{z}_i^{\lim}(t_i))$ satisfying $\sum^{i+1} v_j^2 < p^2$. First of all, since by induction $\underline{z}_i^{\lim}(t_i) \in I_i(\varepsilon_0)$, and since by hypothesis the collision times are separated enough, i.e. $t_{i+1} < t_i - \delta$, Proposition 7.1.1 implies that $\underline{z}_{i+1}^{\lim}(t_{i+1})$ is again a configuration in $I_{i+1}(\varepsilon_0)$, which proves the first part of our induction.

Now, for $\varepsilon > 0$ small enough we have by the induction assumption that

$$\forall \ell \leq i,\ |x_\ell(t_i) - x_\ell^{\lim}(t_i)| \leq \varepsilon(i-1) \leq \varepsilon A^{K+1} \leq a, \tag{115}$$

so that both trajectories are close enough by the condition (106) ordering our parameters. Hence, thanks to the recursion hypothesis $\underline{z}_i^{\lim}(t_i) \in I_i(\varepsilon_0)$, this condition (115) implies by Proposition 7.1.1 that the pseudo-trajectories $\underline{z}_{i+1}$ do not collide at past times, which is also true for the limit trajectories since particles are points. That way, all the velocities remain constant on the interval $]t_{i+1}, t_i]$; since they coincide at time t_i by induction and construction, they coincide on all the interval, which proves the second part of our induction.

Eventually, since the new particle is added at position $x_{i+1}(t_i) = x_{m_i}(t_i) + \varepsilon\omega_{i+1}$, it shifts at most from its limit version by ε, and eventually as the velocities coincide and using one last time the recursion hypothesis we get

$$\forall \ell \leq i+1,\ \forall u \in]t_{i+1}, t_i],\ |x_\ell(u) - x_\ell^{\lim}(u)| \leq \varepsilon(i-1) + \varepsilon \leq \varepsilon i, \tag{116}$$

which concludes the proof by continuity of the trajectories in the space variable.

□

8 Successive approximations (end of the proof of the convergence)

After the pruning of Section 6, we are brought back to studying $F_1^{[K]} - g_1^{[K]}$. In this section, we will quantify the errors made to get non-recollisioning trajectories and hence conclude the proof of Theorem 5.1.

8.1 Energy truncation

We start by the error due to the energy truncation. For any $p > 0$, using the notation

$$\sum_{\underline{j}} a_{\underline{j}} \doteq \sum_{j_1}^{A-1} \cdots \sum_{j_K}^{A^K-1} a_{\underline{j}},$$

we define

$$F_1^{[K,p]} = \sum_{\underline{j}} \alpha^{J_K-1} \sum_{\underline{m}\in\mathcal{M}_{\underline{j}}} f_{\underline{j},\underline{m}}^{[K,p]} \tag{117}$$

where similarly to (101), adding the truncation condition,

$$f_{\underline{j},\underline{m}}^{[K,p]}(t,z_1) = \int_{T_{\underline{j}}^h(t)} \mathrm{d}\underline{t} \int \mathrm{d}\underline{v}\mathrm{d}\underline{\omega} \prod_{i=1}^{J_K-1} \Big[(v_{i+1} - v_{m_i})\cdot\omega_{i+1}\Big] \mathbb{1}_{\left[H_k(\underline{z}_{J_k}(0))\le p^2/2\right]} F_{J_K}\big(0,\underline{z}_{J_k}(0)\big). \tag{118}$$

We define similarly the truncated limit functions $g_{\underline{j},\underline{m}}^{[K,p]}$ as in (102). We have the following estimate on the error made by truncating the velocities.

Proposition 8.1.1 (Control of the energy truncation error). *There is a constant $C_{d,\beta}$ depending only on the dimension and the temperature such that the following bound holds*

$$\left|\left|F_1^{[K]} - F_1^{[K,p]}\right|\right|_{\mathbb{L}^\infty([0,t]\times\mathcal{D}^d)} + \left|\left|g^{[K]} - g^{[K,p]}\right|\right|_{\mathbb{L}^\infty([0,t]\times\mathcal{D}^d)} \le \|\rho\|_\infty A^{K(K+1)}(\alpha C_{d,\beta} t)^{A^{K+1}} e^{-\frac{\beta}{4}p^2}.$$

Proof. Calculus is the same for the BBGKY and for the limiting hierarchies. Let us hence deal with the first case. Since kinetic energy (92) depends only on the choice of added velocities $\underline{v} = (v_2,\dots,v_{J_K})$, the indicator function does not depend on $\underline{m}$ and is thus compatible with the original formulation in terms of successive-collision operators (98). The same argument that in the proof of Proposition 6.3.1 allows to merge the operators when taking the modulus, so that eventually one gets

$$\begin{aligned}\left|\left|\sum_{\underline{m}\in\mathcal{M}_{\underline{j}}} (f_{\underline{j},\underline{m}}^{[K,p]} - f_{\underline{j},\underline{m}}^{[K]})\right|\right|_{\mathbb{L}^\infty} &\le \left|\left||Q|_{1,J_K}(t)\left(\mathbb{1}_{\left[H_{J_K}\left(\underline{v}_{J_K}\right)\ge p^2/2\right]} F_{J_K}(0)\right)\right|\right|_{\mathbb{L}^\infty} \\ &\le \left(\frac{Ct}{(\beta/2)^{(d+1)/2}}\right)^{J_K-1} \left|\left|\mathbb{1}_{\left[H_{J_K}\left(\underline{v}_{J_K}\right)\ge p^2/2\right]} F_{J_K}(0)\right|\right|_{\varepsilon,J_K,\beta/2},\end{aligned}$$

using the continuity estimate of Proposition 6.1.3. Now, the definition of the weighted norms and the estimate (89), we have

$$\begin{aligned}\left|\left|\mathbb{1}_{\left[H_{J_K}\left(\underline{v}_{J_K}\right)\ge p^2/2\right]} F_{J_K}(0)\right|\right|_{\varepsilon,J_K,\beta/2} &\le \sup\left|\mathbb{1}_{\left[H_{J_K}\,\ge\, p^2/2\right]} e^{\beta H_{J_K}} e^{-\frac{\beta}{2}H_{J_K}} e^{+\frac{\beta}{4}p^2} e^{-\frac{\beta}{4}p^2} F_{J_K}(0)\right| \\ &\le ||F_{J_K}(0)||_{\varepsilon,J_K,\beta}\, e^{-\frac{\beta}{4}p^2} \\ &\le \left(\frac{\beta}{2\pi}\right)^{\frac{d}{2}} \|\rho\|_{\mathbb{L}^\infty} e^{-\frac{\beta}{4}p^2},\end{aligned}$$

so that summing over $\underline{j}$ we get

$$\left|\left|F_1^{[K,p]} - F_1^{[K]}\right|\right|_{\mathbb{L}^\infty} \leq \|\rho\|_{\mathbb{L}^\infty} \sum_{\underline{j}} (\alpha C_{d,\beta} t)^{J_K - 1} e^{-\frac{\beta}{4}p^2}$$
$$\leq \|\rho\|_{\mathbb{L}^\infty} A^{K(K+1)} (\alpha C_{d,\beta} t)^{A^{K+1}} e^{-\frac{\beta}{4}p^2},$$

using – like in the proof of Proposition 6.3.1 estimating the pruned-out term – that $J_K \leq A^{K+1}$, and hence conluding the proof.

□

8.2 Time separation

We now estimate the error due the time separation. Like in the previous section, let us define

$$F_1^{[K,p,\delta]} = \sum_{\underline{j}} \alpha^{J_K - 1} \sum_{\underline{m} \in \mathcal{M}_{\underline{j}}} f_{\underline{j},m}^{[K,p,\delta]} \tag{119}$$

where similarly to (118), adding the separation condition encoded in the time set $T_{\underline{j}}^{h,\delta}(t)$ (110),

$$f_{\underline{j},m}^{[K,p,\delta]}(t, z_1) = \int_{T_{\underline{j}}^{h,\delta}(t)} \mathrm{d}\underline{t} \int \mathrm{d}\underline{v}\mathrm{d}\underline{\omega} \prod_{i=1}^{J_K - 1} \left[(v_{i+1} - v_{m_i}) \cdot \omega_{i+1}\right] \mathbb{1}_{\left[H_k(\underline{z}_{J_k}(0)) \leq p^2/2\right]} F_{J_K}\left(0, \underline{z}_{J_k}(0)\right), \tag{120}$$

and similarly the limit version $g^{[K,p,\delta]}$. Then we have the following proposition.

Proposition 8.2.1 (Control of the time separation error)**.** *There is a constant $C_{d,\beta}$ depending only on the dimension and the temperature such that the following bound holds*

$$\left|\left|F_1^{[K,p]} - F_1^{[K,p,\delta]}\right|\right|_{\mathbb{L}^\infty([0,t] \times \mathcal{D}^d)} + \left|\left|g^{[K,p]} - g^{[K,p,\delta]}\right|\right|_{\mathbb{L}^\infty([0,t] \times \mathcal{D}^d)} \leq \delta A^{3K^2} (\alpha C_{d,\beta} t)^{A^{K+1}} ||\rho||_{\mathbb{L}^\infty}.$$

Proof. Once again, the proof is similar for the BBGKY and limit cases, and we prove it in the first case. At fixed $\underline{j}$ and $\underline{m} \in \mathcal{M}_{\underline{j}}$, the difference $f_{\underline{j},m}^{[K,p]} - f_{\underline{j},m}^{[K,p,\delta]}$ is the sum of $J_K - 1$ integrals over two chosen consecutive times closer than δ. Recalling the proof of the continuity estimate (Proposition 6.1.3), where the integration in time was providing a coefficient $t^{J_K - 1}/(J_K - 1)!$, here it can be improved into a factor $\delta t^{J_K - 2}/(J_K - 2)!$ – so that the multiplicative difference being a factor $(J_K - 1)\delta/t$, and since we are summing $J_K - 1$ similar integrals, we get here

$$\left|\left|\sum_{\underline{m} \in \mathcal{M}_{\underline{j}}} (f_{\underline{j},\underline{m}}^{[K,p]} - f_{\underline{j},\underline{m}}^{[K,p,\delta]})\right|\right|_{\mathbb{L}^\infty} \leq (J_K - 1)\delta C_{d,\beta}^{J_K - 1} (J_K - 1) t^{J_K - 2} ||\rho||_{\mathbb{L}^\infty},$$

and hence once again using the fact that $J_K - 1 \leq A^{K+1}$ and summing, we get

$$\left|\left|F_1^{[K,p]} - F_1^{[K,p,\delta]}\right|\right|_{\mathbb{L}^\infty} \leq A^{(K+1)(K+2)} (\alpha C_{d,\beta})^{A^{K+1}} \delta t^{A^{K+1}} ||\rho||_{\mathbb{L}^\infty}, \tag{121}$$

concluding the proof by noticing that as soon as $K \geq 2$ the following holds: $(K+1)(K+2) \leq 3K^2$.

□

8.3 Restriction to non-pathological collision parameters

Finally, we have to estimate the error do to the restriction to non-pathological collision parameters in the inductive construction of Proposition 7.2.1. For the last time then, we introduce

$$\widetilde{F}_1^{[K,p,\delta]} = \sum_{\underline{j}} \alpha^{J_K-1} \sum_{\underline{m}\in\mathcal{M}_{\underline{j}}} \widetilde{f}_{\underline{j},\underline{m}}^{[K,p,\delta]} \tag{122}$$

where similarly to (120), adding the restriction to $\Pi(z_1,\underline{j},\underline{t},\underline{m})^c$ (113),

$$\widetilde{f}_{\underline{j},\underline{m}}^{[K,p,\delta]}(t,z_1) = \int_{T_{\underline{j}}^{h,\delta}(t)} \mathrm{d}\underline{t} \int_{\Pi(z_1,\underline{j},\underline{t},\underline{m})^c} \mathrm{d}\underline{v}\mathrm{d}\underline{\omega} \prod_{i=1}^{J_K-1} \left[(v_{i+1}-v_{m_i})\cdot\omega_{i+1}\right] \mathbb{1}_{\left[H_k(\underline{z}_{J_k}(0))\leq p^2/2\right]} F_{J_K}\big(0,\underline{z}_{J_k}(0)\big),$$

and similarly at the limit $\widetilde{g}^{[K,p,\delta]}$. We then have the following estimate on the error made by doing so.

Proposition 8.3.1 (Control of the non-recollisioning error)**.** *There is a constant $C_{d,\beta}$ depending only on the dimension and the temperature such that the following bound holds*

$$\begin{aligned}\left|\left|F_1^{[K,p,\delta]} - \widetilde{F}_1^{[K,p,\delta]}\right|\right|_{\mathbb{L}^\infty([0,t]\times\mathcal{D}^d)} &+ \left|\left|g^{[K,p,\delta]} - \widetilde{g}^{[K,p,\delta]}\right|\right|_{\mathbb{L}^\infty([0,t]\times\mathcal{D}^d)} \\ &\leq A^{3K^2}(C_{d,\beta}\alpha t)^{A^{K+1}}\left(\eta^d + p^d\left(\frac{a}{\varepsilon_0}\right)^{\frac{d-1}{2}} + p^{\frac{d+1}{2}}\left(\frac{\varepsilon_0}{\delta}\right)^{\frac{d-1}{2}}\right)\|\rho\|_\infty.\end{aligned}$$

Proof. The proof is very similar to the two previous ones, the key here is to estimate the volume of the set of pathological sequences of collision parameters. Recalling its definition (113) and applying Proposition 7.1.1 to control the successive pathological sets, we get

$$\begin{aligned}|\Pi(z_1,\underline{j},\underline{t},\underline{m})| &\leq \sum_{k=1}^{J_K-1} |\Pi_k^{m_k}(\underline{z}_k^{\lim})| \\ &\leq (J_K-1)C(J_K-1)\left(\eta^d + p^d\left(\frac{a}{\varepsilon_0}\right)^{\frac{d-1}{2}} + p^{\frac{d+1}{2}}\left(\frac{\varepsilon_0}{\delta}\right)^{\frac{d-1}{2}}\right),\end{aligned}$$

which allows to conclude the proof, once again thanks to the inequality $J_K - 1 \leq A^{K+1}$.

□

Now that we have constructed approximations of our distributions that avoid recollisions, we can at last compare both the BBGKY and limiting distributions with the following proposition, thanks to the coupled pseudo-trajectories.

Proposition 8.3.2 (Proximity of non-recollisioning distributions)**.** *There is a constant $C_{d,\beta}$ depending only on the dimension and the temperature such that the following bound holds for ε small enough*

$$\left|\left|\widetilde{F}_1^{[K,p,\delta]} - \widetilde{g}^{[K,p,\delta]}\right|\right|_{\mathbb{L}^\infty([0,t]\times\mathcal{D}^d)} \leq A^{K(K+1)}(C_{d,\beta}\alpha t)^{A^{K+1}}\|\rho\|_{\mathbb{L}^\infty}\alpha\varepsilon. \tag{123}$$

Proof. Within the formulation in terms of pseudo-trajectories, the only difference between both terms we want to compare is the initial distributions of the trajectories $F_{J_K}\big(0,\underline{z}_{J_k}(0)\big)$ and $g_{J_K}\big(0,\underline{z}_{J_k}^{\lim}(0)\big)$ – as one can see for example between (101) and (102). Since x_1 is the only position on which depends

g_{J_K}, and since *by the inductive construction* of Proposition 7.2.1 the velocities are the same in both coupled pseudo-trajectories for ε small enough, we have

$$g_{J_K}\big(0, \underline{z}^{\lim}_{J_k}(0)\big) = g_{J_K}\big(0, \underline{z}_{J_k}(0)\big). \tag{124}$$

Now, once again Proposition 7.2.1 asserts that the initial data $\underline{z}_{J_k}(0)$ is in $I_{J_K}(\varepsilon_0/2)$ and hence in $\mathcal{D}^{\varepsilon}_{J_K}$, so that one can apply Proposition 6.1.2 on the proximity of the initial distributions to get

$$\big|g_{J_K}\big(0, \underline{z}_{J_k}(0)\big) - F_{J_K}\big(0, \underline{z}_{J_k}(0)\big)\big| \leq C^{J_K} \|\rho\|_{\mathbb{L}^\infty} \alpha\varepsilon, \tag{125}$$

and the proof ends like the previous ones using the continuity estimates and summing over $\underline{j}$.

□

8.4 Final estimate on the pruned distributions

We synthetize all these errors to have a finale result on the pruned distributions.

Proposition 8.4.1. *In the following scaling*

$$\alpha t \leq (\log\log\mu)^{\frac{A-1}{A}} \ \text{and}\ K = \left\lfloor \frac{\log\log\mu}{2\log A} \right\rfloor, \tag{126}$$

one has this finale estimate between the BBGKY and Boltzmann pruned distributions

$$\left|\left| F_1^{[K]} - g^{[K]} \right|\right|_{\mathbb{L}^\infty([0,t]\times\mathcal{D}^d)} \leq \exp\left(C\sqrt{\log\mu}\log\log\mu\right) \|\rho\|_{\mathbb{L}^\infty} \varepsilon^{\frac{d-1}{2(d+1)}}. \tag{127}$$

Proof. Putting together all previous estimates, first Propositions 8.1.1 for the energy truncation error, 8.2.1 for the time delay error and 8.3.1 for the non-recollisioning error and then Proposition 8.3.2 for the proximity of the non-recollisioning distributions, we get

$$\begin{aligned}\left|\left| F_1^{[K]} - g^{[K]} \right|\right|_{\mathbb{L}^\infty} &\leq \left|\left| F_1^{[K]} - \widetilde{F}_1^{[K,p,\delta]} \right|\right|_{\mathbb{L}^\infty} + \left|\left| \widetilde{F}_1^{[K,p,\delta]} - \widetilde{g}_1^{[K,p,\delta]} \right|\right|_{\mathbb{L}^\infty} + \left|\left| \widetilde{g}_1^{[K,p,\delta]} - g^{[K]} \right|\right|_{\mathbb{L}^\infty} \\ &\leq \|\rho\|_{\mathbb{L}^\infty} A^{3K^2} (C\alpha t)^{A^{K+1}} \left(e^{-\frac{\beta}{4}p^2} + \frac{\delta}{t} + \eta^d + p^d \left(\frac{a}{\varepsilon_0}\right)^{\frac{d-1}{2}} + p^{\frac{p+1}{2}} \left(\frac{\varepsilon_0}{\delta}\right)^{\frac{d-1}{2}} \right) \\ &\quad + A^{3K^2} (C\alpha t)^{A^{K+1}} \|\rho\|_{\mathbb{L}^\infty} \alpha\varepsilon.\end{aligned}$$

Now we have to tune our parameters, that we choose in the following way, satisfying condition (106)

$$a = A^{K+1}\varepsilon \ll \varepsilon_0 = \varepsilon^{\frac{d}{d+1}}, \quad \delta = \varepsilon^{\frac{d-1}{d+1}}, \quad \eta = \varepsilon^{\frac{1}{d+2}}, \quad p = 2\sqrt{\frac{-\log\varepsilon}{\beta}}. \tag{128}$$

Hence, since $A^K \leq \sqrt{\log\mu} = \sqrt{\log\alpha - (d-1)\log\varepsilon}$ and $\alpha t \leq \log\log\mu$, and up to change the constant, we may eventually conclude the proof of the proposition

$$\begin{aligned}\left|\left| F_1^{[K]} - g^{[K]} \right|\right|_{\mathbb{L}^\infty} &\leq A^{3K^2} \|\rho\|_{\mathbb{L}^\infty} (C\alpha t)^{A^{K+1}} \times \\ &\qquad \left(\varepsilon + \varepsilon^{\frac{d-1}{d+1}} + \varepsilon^{\frac{d}{d+2}} + |\log\varepsilon|^{\frac{d}{2}} \left(A^{K+1}\right)^{\frac{d-1}{2}} \varepsilon^{\frac{d-1}{2(d+1)}} + |\log\varepsilon|^{\frac{d+1}{4}} \varepsilon^{\frac{d-1}{2(d+1)}} + \alpha\varepsilon \right) \\ &\leq A^{3K^2} \|\rho\|_{\mathbb{L}^\infty} (C\alpha t)^{A^{K+1}} \left(\varepsilon^{\frac{d-1}{d+1}} + |\log\varepsilon|^{\frac{3d-1}{4}} \varepsilon^{\frac{d-1}{2(d+1)}} \right) \\ &\leq \exp\left(C\sqrt{\log\mu}\log\log\mu\right) \|\rho\|_{\mathbb{L}^\infty} \varepsilon^{\frac{d-1}{2(d+1)}}.\end{aligned}$$

□

8.5 Proof of the theorem

Thanks to the last proposition, under the same assumptions and considering the pruning error controlled by Proposition 6.3.1 we know that in our scaling limit, chosing

$$h = \frac{c\eta_0}{\alpha^{A/(A-1)}t^{1/(A-1)}} \Leftrightarrow \eta_0 = \frac{(\alpha t)^{A/(A-1)}}{cK},$$

we have asymptotically

$$\begin{aligned}||F_1 - g||_{\mathbb{L}^\infty([0,t]\times\mathcal{D}^d} &\leq C\left(\eta_0^A + \exp\left(\sqrt{\log\mu}\log\log\mu\right)\varepsilon^{\frac{d-1}{2(d+1)}}\right)\|\rho\|_{\mathbb{L}^\infty} \\ &\leq \widetilde{C}\left(\frac{(\alpha t)^{A/(A-1)}}{\log\log\mu}\right)^A \|\rho\|_{\mathbb{L}^\infty}.\end{aligned}$$

□

This way we concluded the proof of Theorem 5.1. Concerning further studies, one may want to improve the rate of convergence so as to get hydrodynamic limits such as Theorem 5.2 for larger time scales. Another current active research direction is to precise the probabilistic results exposed in Section 5.2, for example with large deviation results [3].

9 Appendix

9.1 Change of variable on the hypersurface $\Sigma(i,j)$

We hereafter prove the change of variable used in section 2.2 to fulfill the calculus of BBGKY hierarchy in (15), with the notation presented in the concerned section, to change the parametrization of the hypersurface $\Sigma(i,j)$, making appear a factor $\sqrt{2}$.

Let us consider the case of two particles $x = x_i$ and $y = x_j$, since the other ones are free of conditions in $\Sigma(i,j)$. Let us denote $\phi(\theta_1,\ldots,\theta_{d-1})$ a parametrization of $\varepsilon \cdot \mathbb{S}^{d-1}$. We thus have the following parametrization of $\Sigma(i,j) \ni (x,y)$ as a union of balls with centers $x = (t_1,\ldots,t_d)$ and radiuses ε,

$$\Phi(t_1,\ldots,t_d,\theta_1,\ldots,\theta_{d-1}) = \Big(t_1;\ldots;t_d;t_1+\phi(\theta_1,\ldots,\theta_{d-1});\ldots;t_d+\phi(\theta_1,\ldots,t_{d-1})\Big), \tag{129}$$

whose Jacobian is given by

$$J_\Phi = \left(\begin{array}{c|c} I_d & 0 \\ \hline I_d & J_\phi \end{array}\right). \tag{130}$$

Then, by integration over a hypersurface, the element of surface of $\Sigma(i,j)$ is given by

$$\mathrm{d}\sigma(x,y) = A_\Phi(T_d,\Theta_{d-1})\mathrm{d}T_d\mathrm{d}\Theta_{d-1}, \tag{131}$$

where the area $\mathcal{A}_\Phi(T_d,\Theta_{d-1})$ is yielded by the norm of the cross product of the columns of J_Φ, or differently formulated

$$\mathcal{A}_\Phi^2 = \sum_{i=1}^{2d} \Delta_\Phi(k)^2, \tag{132}$$

where $\Delta_\Phi(k)$ is the determinant of J_Φ deprived of its k-th line. A simple computation of matrix calculus using the relation (130) between both Jacobians – developping for $k \leq d$ along the k-th column which has only a single 1 in $(d+k)$-th position – yields

$$\Delta_\Phi(k)^2 = \begin{cases} \Delta_\phi(k)^2 \text{ if } k \leq d \\ \Delta_\phi(k-d)^2 \text{ otherwise,} \end{cases} \tag{133}$$

so that

$$\mathcal{A}_\Phi^2 = 2\mathcal{A}_\phi^2, \tag{134}$$

i.e. eventually

$$\mathrm{d}\sigma(x,y) = \sqrt{2}\cdot \mathrm{d}x\mathrm{d}\omega(y). \tag{135}$$

□

9.2 Change of variable from pre- to post-collisional velocities

At fixed ω, the map $(v,v_c) \mapsto (v^*,{v_c}^*)$ has Jacobian 1. Indeed, denoting $\Omega_2 = \omega \times {}^t\omega$, we have

$$J = \left(\begin{array}{c|c} I-\Omega_2 & \Omega_2 \\ \hline \Omega_2 & I-\Omega_2 \end{array}\right), \tag{136}$$

so that – inspired by the scalar case, conjugate by $\left(\begin{array}{c|c} I & I \\ \hline I & -I \end{array}\right)$, and then diagonalize Ω_2 of rank 1 with the eigenpair $(|\omega|^2,\omega)$,

$$\det J = \det(I)\cdot\det(I-2\Omega_2) = 1-2|\omega|^2 = -1. \tag{137}$$

□

9.3 Asymptotic study of the partition functions

This appendix is dedicated to the study of the canonical and grand-canonical partition functions, defined respectively in (32) and (23). In particular, it justifies the fact that the random grand canonical number of particles $\mathcal{N}$ (31) gets close to a Poisson variable when μ goes to infinity in the Boltzmann-Grad scaling.

9.3.1 Canonical partition functions

First of all, at fixed $N \in \mathbb{N}^*$, the Lebesgue's dominated convergence theorem asserts that

$$\mathcal{Z}_N^c(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 1. \tag{138}$$

This is what is needed on the canonical partition function to justify the asymptotic behavior of $\mathcal{N}$.

Nevertheless, one may want to know what happens to them in the Boltzmann-Grad limit (i.e. such that $N\varepsilon^{d-1} = 1$). Thanks to the exclusion condition of $\mathcal{D}_N^\varepsilon$, one can be sure that the conditions $|x_i - x_j| > \varepsilon/2$ are all disjoint. Since $\varepsilon/2$ is the radius of the spheres, a way to see these conditions is to say that the *center* of a particle cannot enter another particle. Hence, taking this small margin we may integrate over these disjoint conditions to get, integrating over x_N and then iterating,

$$\begin{aligned}\mathcal{Z}_N^c &= \int_{\mathbb{T}^{dp}} \prod_{i\neq j} \mathbb{1}_{|x_i-x_j|>\varepsilon} \mathrm{d}\underline{x}_N \\ &\leq \left(1-(N-1)|\mathcal{B}_d|\frac{\varepsilon^d}{2^d}\right)\int_{\mathbb{T}^{d(N-1)}} \prod_{i\neq j} \mathbb{1}_{|x_i-x_j|>\varepsilon} \mathrm{d}\underline{x}_{N-1} \\ &\leq \prod_{i=1}^{N-1}\left(1-i|\mathcal{B}_d|\frac{\varepsilon^d}{2^d}\right),\end{aligned}$$

so that

$$\log \mathcal{Z}_N^c \leq \sum_{i=1}^{N-1} \log\left(1-i|\mathcal{B}_d|\frac{\varepsilon^d}{2^d}\right), \tag{139}$$

and by the concavity inequality $\log(1-x) \leq -x$, we have for $d \geq 3$ that

$$\log \mathcal{Z}_N^c \leq \sum_{i=1}^{N-1}\left(-i|\mathcal{B}_d|\frac{\varepsilon^d}{2^d}\right) = -|\mathcal{B}_d|\frac{\varepsilon^d}{2^d}\frac{N(N-1)}{2} = -\frac{|\mathcal{B}_d|}{2^{d+1}}\varepsilon(N-1) \xrightarrow[N\to\infty]{} -\infty, \tag{140}$$

in the Boltzmann-Grad scaling, so that eventually for $d \geq 3$,

$$\mathcal{Z}_N^c \xrightarrow[N\to\infty]{N^{-1}=\varepsilon^{d-1}} 0.$$

This is an interesting phenomenon: as μ goes to infinity, the probability $\mathbb{P}[\mathcal{N} = N]$ at fixed N will behave asymptotically like $e^{-\mu}\dfrac{\mu^N}{N!}$ (see the following study), but for N around the expectation μ of the Poisson law – where the mass is concentrated – this behavior will collapse.

9.3.2 Cumulants

So as to study the grand canonical partition function, we introduce some combinatorial tools called *cumulants*, which allows to decompose the exclusion condition into clusters in the spirit of Poincaré's inclusion–exclusion principle. Most of the following results are proved in [3]. We denote $\mathcal{P}_n$ the set of partitions of $[\![1,n]\!]$ and $\mathcal{P}_n^s$ the set of partitions into s parts. For a partition $\sigma = (\sigma_1, \cdots, \sigma_s) \in \mathcal{P}_n^s$, we denote $\underline{z}_{\sigma_j} = (z_{\sigma_j(1)}, \cdots, z_{\sigma_j(|\sigma_j|)})$.

The *cumulants* are thus defined in the following way

$$\varphi_n(\underline{z}_n) = \sum_{s=1}^{n} \sum_{\sigma \in \mathcal{P}_n^s} (-1)^{s+1}(s-1)! \prod_{j=1}^{s} \mathbb{1}_{\mathcal{D}^{\varepsilon}_{|\sigma_j|}}(\underline{z}_{\sigma_j}), \tag{141}$$

and one may find back the indicator function of the exclusion thanks to the following inversion formula

$$\mathbb{1}_{\mathcal{D}_n^{\varepsilon}}(\underline{z}_n) = \sum_{s=1}^{n} \sum_{\sigma \in \mathcal{P}_n^s} \prod_{j=1}^{s} \varphi_{|\sigma_j|}(\underline{z}_{\sigma_j}). \tag{142}$$

Finally, denoting $\mathcal{G}_n$ the set of connected graphs on $[\![1,n]\!]$ and $\mathcal{T}_n$ the set of connected trees, we have the following relation

$$\varphi_n(\underline{z}_n) = \sum_{G \in \mathcal{G}_n} \prod_{\{i,j\} \in E(G)} (-\mathbb{1}_{|x_i - x_j| \leq \varepsilon}) \tag{143}$$

and the ensuing *tree inequality*

$$|\varphi_n(\underline{z}_n)| \leq \sum_{T \in \mathcal{T}_n} \prod_{\{i,j\} \in E(T)} \mathbb{1}_{|x_i - x_j| \leq \varepsilon}(\underline{z}_n). \tag{144}$$

9.3.3 Grand canonical partition function

With the notation of previous section, we can write like in [11] that

$$\begin{aligned}
\mathcal{Z} &= 1 + \sum_{p \geqslant 1} \frac{\mu^p}{p!} \sum_{s=1}^{p} \sum_{\sigma \in \mathcal{P}_p^s} \int \prod_{k=1}^{s} \varphi_{|\sigma_k|}(\underline{z}_{\sigma_k}) \mathrm{d}\underline{z}_p \\
&= 1 + \sum_{p \geqslant 0} \frac{\mu^p}{p!} \sum_{s=1}^{p} \sum_{\substack{k_1, \cdots, k_s \geqslant 1 \\ \sum k_j = p}} \frac{1}{s!} \binom{p}{k_1} \binom{p-k_1}{k_2} \cdots \binom{p-k_1-\cdots-k_{s-2}}{k_{s-1}} \prod_{i=1}^{s} \int \varphi_{k_i}(\underline{z}_{k_i}) \mathrm{d}\underline{z}_{k_i} \\
&= 1 + \sum_{s \geqslant 1} \frac{1}{s!} \prod_{i=1}^{s} \sum_{k_1 \geqslant 1} \frac{\mu^{k_i}}{k_i!} \int \varphi_{k_i} \\
&= 1 + \sum_{s \geqslant 1} \frac{1}{s!} \left(\sum_{k \geqslant 1} \frac{\mu^k}{k!} \int \varphi_k \right)^s \\
&= \exp\left(\sum_{k \geqslant 1} \frac{\mu^k}{k!} \int \varphi_k \right).
\end{aligned}$$

One can check from Ruelle's book [8] that the latter series is alternating, so that thanks to the tree inequality (144), we get

$$\mathcal{Z} = \exp(\mu + o(1)). \tag{145}$$

This is the second argument used to justify that $\mathcal{N}$ asymptotically gets closer to a Poisson variable of parameter μ.

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